HARMONIC ANALYSIS ON THE SU(2) DYNAMICAL QUANTUM GROUP

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ABSTRACT. Dynamical quantum groups were recently introduced by Etingof and Varchenko as an algebraic framework for studying the dynamical Yang–Baxter equation, which is precisely the Yang–Baxter equation satisfied by 6j-symbols. We investigate one of the simplest examples, generalizing the standard SU(2) quantum group. The matrix elements for its corepresentations are identified with Askey–Wilson polynomials, and the Haar measure with the Askey–Wilson measure. The discrete orthogonality of the matrix elements yield the orthogonality of q-Racah polynomials (or quantum 6j-symbols). The Clebsch–Gordan coefficients for representations and corepresentations are also identified with q-Racah polynomials. This results in new algebraic proofs of the Biedenharn–Elliott identity satisfied by quantum 6j-symbols.

1. INTRODUCTION

Quantum groups first arose in the 1980's as an algebraic framework for studying R-matrices, which have their origin in statistical mechanics. An R-matrix is a solution of the Yang–Baxter equation, which exists in several versions. While the simplest examples of quantum groups are constructed from constant solutions of the Yang–Baxter equation, the R-matrices of statistical mechanics usually depend on external parameters. For vertex models, these are known as spectral parameters, while for face models so called dynamical parameters are present.

The fundamental Faddeev–Reshetikhin–Sklyanin–Takhtajan (FRST) construction assigns a bialgebra (and in many cases a Hopf algebra) to any constant solution of the quantum Yang–Baxter equation. Generalizations of this construction to R-matrices with spectral parameters lead to Yangians, quantum affine algebras and Sklyanin algebras, depending on whether the R-matrix is a rational, a trigonometric or an elliptic function. In [FV], Felder and Varchenko gave a similar construction starting from an elliptic R-matrix involving both spectral and dynamical parameters. Motivated by this example, Etingof and Varchenko [EV2, EV3] have developed an algebraic framework for studying dynamical R-matrices. The resulting "dynamical quantum groups" are not Hopf algebras, but rather Hopf algebroids. The appearance of "oids" is reflected in the correspondence between quasiclassical limits of dynamical R-matrices and Poisson structures on Lie groupoids discovered in [EV1].

In the present paper we study one of the simplest examples of dynamical quantum groups, constructed from a trigonometric dynamical R-matrix. In particular, we are interested in the special functions related to its representation theory. It turns out that fundamental objects such as matrix elements and Clebsch–Gordan coefficients can be identified with q-Racah polynomials (or quantum 6j-symbols) and

Askey–Wilson polynomials. This results in new algebraic proofs of the orthogonality relation and Biedenharn–Elliott identity satisfied by quantum $6j$-symbols. Moreover, we can interpret the orthogonality measure of the Askey–Wilson polynomials as a Haar measure on the dynamical quantum group. To obtain these results we must extend the algebraic machinery introduced by Etingof and Varchenko in several ways, with new definitions and new results. We hope that the present case study will be useful when investigating Felder's elliptic quantum groups [F, FV], and relating their representation theory to the elliptic hypergeometric series introduced in [FT]. One would likewise expect connections between higher rank dynamical quantum groups and multivariable orthogonal polynomials.

Let us recall the definition of the quantum dynamical Yang–Baxter (QDYB) equation, also known as the Gervais–Neveu–Felder equation. Let $\mathfrak{h}$ be a finite-dimensional complex vector space, viewed as a commutative Lie algebra, and $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ a diagonalizable $\mathfrak{h}$-module. In the context of dynamical quantum groups, $\mathfrak{h}$ will typically be a Cartan subalgebra of the corresponding Lie algebra. The QDYB equation may be written as

$$R^{12}(\lambda - h^{(3)})R^{13}(\lambda)R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda)R^{13}(\lambda - h^{(2)})R^{12}(\lambda).$$

This is an identity in the algebra of meromorphic functions $\mathfrak{h}^* \to \operatorname{End}(V \otimes V \otimes V)$. Here $R : \mathfrak{h}^* \to \operatorname{End}(V \otimes V)$ is a meromorphic function, h indicates the action of $\mathfrak{h}$, and the upper indices refer to the factors in the tensor product. For instance, $R^{12}(\lambda - h^{(3)})$ denotes the operator

$$R^{12}(\lambda - h^{(3)})(u \otimes v \otimes w) = (R(\lambda - \mu)(u \otimes v)) \otimes w, \quad w \in V_\mu.$$

A dynamical R-matrix is by definition a solution of the QDYB equation which is $\mathfrak{h}$-invariant, that is, $R : \mathfrak{h}^* \to \operatorname{End}_{\mathfrak{h}}(V \otimes V)$. For an introduction to the QDYB equation and its relation to other topics we refer to [ES].

In the form given above, the QDYB equation first appeared in [GN]. Felder [F] pointed out its equivalence to the star-triangle relation satisfied by the Boltzmann weights of face models. It is also equivalent to one of the classical identities for the $6j$-symbols of quantum mechanics, which reflects the symmetries of the $9j$-symbol [EV3, N]. In this context, the QDYB equation (for $\mathfrak{h} = \mathbb{C}$) goes back to Wigner's 1940 paper [W] (cf. equation (26a) there).

In the example that we will study $\mathfrak{h}$ is one-dimensional, and may be viewed as a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. We identify $\mathfrak{h} = \mathfrak{h}^* = \mathbb{C}$ and take V to be the two-dimensional $\mathfrak{h}$-module $V = \mathbb{C}e_1 \oplus \mathbb{C}e_{-1}$. In the basis $e_1 \otimes e_1$, $e_1 \otimes e_{-1}$, $e_{-1} \otimes e_1$, $e_{-1} \otimes e_{-1}$, the dynamical R-matrix we will consider is given by

$$(1.1) \qquad R(\lambda) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \frac{q^{-1}-q}{q^{2(\lambda+1)}-1} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-2(\lambda+1)}-1} & \frac{(q^{2(\lambda+1)}-q^2)(q^{2(\lambda+1)}-q^{-2})}{(q^{2(\lambda+1)}-1)^2} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

This is the R-matrix arising from $6j$-symbols of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}(2))$, evaluated in the two-dimensional representation [ES]. It can also be interpreted as the

R-matrix for a quasi-Hopf algebra, which can be obtained from $\mathcal{U}_q(\mathfrak{sl}(2))$ via a Drinfel'd twist [B, BBB]. Babelon's construction of the twisting operator uses certain "shifted boundaries" in the quantum algebra. By contrast to the interpretation in terms of $6j$-symbols, this construction collapses in the limit $q \to 1$. The shifted boundaries were rediscovered in [R], where they appear as q-analogues of group elements. This gives a link between the QDYB equation and harmonic analysis with respect to twisted primitive elements of $\mathcal{U}_q(\mathrm{sl}(2))$. On the level of special functions, the results of this paper are largely parallel to those obtained via twisted primitive elements, cf. [GZ, Ko1, Ko2, KV1, K3, NM, R]. However, the conceptual connection between these two approaches remains to be investigated.

We will now briefly summarize the contents of the paper. In §2 we review the generalized FRST construction from [EV2]. We then describe the dynamical quantum group $\mathcal{F}_R(\mathrm{SL}(2))$ which is obtained from the R-matrix (1.1) through this construction. In §3 we introduce finite-dimensional corepresentations of $\mathcal{F}_R(\mathrm{SL}(2))$. The main result of this section is Theorem 3.5, where the matrix elements of our corepresentations are expressed in terms of Askey–Wilson polynomials. In §4 we introduce a family of infinite-dimensional representations of $\mathcal{F}_R(\mathrm{SL}(2))$, and use them to obtain the orthogonality of q-Racah polynomials as discrete orthogonality relations for the matrix elements. In §5 and §6 we consider tensor product decompositions of corepresentations and representations, respectively. In both cases we obtain q-Racah polynomials as Clebsch–Gordan coefficients. This gives new algebraic proofs of the Biedenharn–Elliott identity, which plays a fundamental role in quantum mechanics and is also a master identity from the viewpoint of special functions. In §7 we show that there is a natural Haar functional on our algebra, and that it can be identified with the Askey–Wilson measure.

As was mentioned above, we have to extend the algebraic machinery of Etingof and Varchenko in several ways. Since some readers may be mainly interested in these parts of the paper, we will indicate where they can be found. Our definition of antipode, Definition 2.1, differs from the one given in [EV2]. With our modified definition, we can extend some basic results for Hopf algebras to the present situation, cf. Proposition 2.2 and Lemma 2.9. These are proved in Appendix 1. In §2.2 we give a straight-forward definition of $*$-structure on an $\mathfrak{h}$-algebra. In §3.1 we introduce the concept of corepresentation of an $\mathfrak{h}$-bialgebroid. Instead of unitarity of corepresentations, we speak of unitarizability, cf. Definition 3.11.

To discuss tensor products of corepresentations, we must introduce several new algebraic concepts, cf. §5.1–5.2. Recall that if A and B are Hopf algebras, then so is $A \otimes B$. This is not true for the $\mathfrak{h}$-Hopf algebroids which we study: there is then one kind of tensor product (denoted $\widetilde{\otimes}$ and introduced in [EV2]) which inherits the algebra structure and another one (denoted $\widehat{\otimes}$ and introduced in §5.1) which inherits the coalgebroid structure. The innocent-looking Lemma 5.2 is a key result which relates these structures. Finally, from §7 it is clear what the natural definition of Haar functional on an $\mathfrak{h}$-bialgebroid should be.

2. Preliminaries on dynamical quantum groups

2.1. **Dynamical R-matrices and $\mathfrak{h}$-bialgebroids.** In this section we review some of the results of [EV2]. First we recall the fundamental notions of $\mathfrak{h}$-algebra, $\mathfrak{h}$-bialgebroid and $\mathfrak{h}$-Hopf algebroid. These structures are related to the more general Hopf algebroids introduced by Lu [L]. We then recall the generalized FRST construction, associating an $\mathfrak{h}$-bialgebroid to any dynamical R-matrix.

Throughout this section, $\mathfrak{h}$ will be a finite-dimensional complex commutative Lie algebra and $M_{\mathfrak{h}^*}$ will denote the field of meromorphic functions on the dual of $\mathfrak{h}$.

An $\mathfrak{h}$-*algebra* is a complex associative algebra A with 1, which is bigraded over $\mathfrak{h}^*$, $A = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} A_{\alpha\beta}$, and equipped with two algebra embeddings μ_l, $\mu_r : M_{\mathfrak{h}^*} \to A_{00}$ (the left and right *moment maps*), such that

$$(2.1) \qquad \mu_l(f)a = a\,\mu_l(T_\alpha f), \qquad \mu_r(f)a = a\,\mu_r(T_\beta f), \quad a \in A_{\alpha\beta}, \ f \in M_{\mathfrak{h}^*},$$

where T_α denotes the automorphism $T_\alpha f(\lambda) = f(\lambda + \alpha)$ of $M_{\mathfrak{h}^*}$. A morphism of $\mathfrak{h}$-algebras is an algebra homomorphism preserving the moment maps (and thus also the bigrading).

The *matrix tensor product* $A \widetilde{\otimes} B$ of two $\mathfrak{h}$-algebras is the $\mathfrak{h}^*$-bigraded vector space with

$$(A \widetilde{\otimes} B)_{\alpha\beta} = \bigoplus_\gamma (A_{\alpha\gamma} \otimes_{M_{\mathfrak{h}^*}} B_{\gamma\beta}),$$

where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations

$$(2.2) \qquad \mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b, \quad a \in A, \ b \in B, \ f \in M_{\mathfrak{h}^*}.$$

It follows that

$$(2.3) \qquad a\mu_r^A(f) \otimes b = a \otimes b\mu_l^B(f)$$

in $A \widetilde{\otimes} B$. The multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{A \widetilde{\otimes} B}(f) = \mu_l^A(f) \otimes 1, \qquad \mu_r^{A \widetilde{\otimes} B}(f) = 1 \otimes \mu_r^B(f)$$

make $A \widetilde{\otimes} B$ into an $\mathfrak{h}$-algebra.

We denote by $D_{\mathfrak{h}}$ the algebra of difference operators on $M_{\mathfrak{h}^*}$, consisting of operators

$$\sum_i f_i\, T_{\beta_i}, \quad f_i \in M_{\mathfrak{h}^*}, \ \beta_i \in \mathfrak{h}^*.$$

This is an $\mathfrak{h}$-algebra with the bigrading defined by $f\,T_{-\beta} \in (D_{\mathfrak{h}})_{\beta\beta}$ and both moment maps equal to the natural embedding. For any $\mathfrak{h}$-algebra A, there are canonical $\mathfrak{h}$-algebra isomorphisms $A \simeq A \widetilde{\otimes} D_{\mathfrak{h}} \simeq D_{\mathfrak{h}} \widetilde{\otimes} A$, defined by

$$(2.4) \qquad x \simeq x \otimes T_{-\beta} \simeq T_{-\alpha} \otimes x, \quad x \in A_{\alpha\beta}.$$

Thus the algebra $D_{\mathfrak{h}}$ plays the role of unit object in the category of $\mathfrak{h}$-algebras.

An $\mathfrak{h}$-*bialgebroid* is an $\mathfrak{h}$-algebra A equipped with two $\mathfrak{h}$-algebra homomorphisms, $\Delta : A \to A \widetilde{\otimes} A$ (the coproduct) and $\varepsilon : A \to D_{\mathfrak{h}}$ (the counit), such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and, under the identifications (2.4), $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$.

We will also need the concept of an $\mathfrak{h}$-*Hopf algebroid*. Our definition differs slightly from the one given in [EV2].

Definition 2.1. *An $\mathfrak{h}$-Hopf algebroid is an $\mathfrak{h}$-bialgebroid A equipped with a $\mathbb{C}$-linear map $S : A \to A$, called the antipode, such that*

$$(2.5) \qquad S(\mu_r(f)a) = S(a)\mu_l(f), \quad S(a\mu_l(f)) = \mu_r(f)S(a), \quad a \in A, \ f \in M_{\mathfrak{h}^*},$$

$$(2.6) \qquad \begin{aligned} m \circ (\mathrm{id} \otimes S) \circ \Delta(a) &= \mu_l(\varepsilon(a)1), \quad a \in A, \\ m \circ (S \otimes \mathrm{id}) \circ \Delta(a) &= \mu_r(T_\alpha(\varepsilon(a)1)), \quad a \in A_{\alpha\beta}, \end{aligned}$$

where m denotes multiplication and where $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{\mathfrak{h}^*}$.

The conditions (2.5) guarantee that the left-hand sides of (2.6) are well-defined; cf. Lemma A.3 in Appendix 1. Note that since $\varepsilon(A_{\alpha\beta}) = 0$ for $\alpha \neq \beta$, the translation T_α in (2.6) can be replaced by T_β. In [EV2], the translation T_α is missing from (2.6), and (2.5) is replaced by the stronger property that S is an algebra antihomomorphism which interchanges the moment maps. This is a consequence of our definition.

Proposition 2.2. *The antipode of an $\mathfrak{h}$-Hopf algebroid is unique. Moreover, it satisfies*

$$(2.7) \qquad\qquad\qquad S(A_{\alpha\beta}) \subseteq A_{-\beta,-\alpha},$$

$$(2.8) \ \ S(ab) = S(b)S(a), \quad \Delta \circ S = \sigma \circ (S \otimes S) \circ \Delta, \quad S(1) = 1, \quad \varepsilon \circ S = S^{D_\mathfrak{h}} \circ \varepsilon,$$

$$(2.9) \qquad\qquad S(\mu_l(f)) = \mu_r(f), \quad S(\mu_r(f)) = \mu_l(f),$$

where σ is the flip $\sigma(a \otimes b) = b \otimes a$ and where $S^{D_\mathfrak{h}}$ is the algebra antihomomorphism of $D_\mathfrak{h}$ defined by $S^{D_\mathfrak{h}}(f\,T_\alpha) = T_{-\alpha} \circ f = (T_{-\alpha}f)T_{-\alpha}$.

Moreover, if A is generated as an algebra by a subset X, and S is an algebra antihomomorphism of A such that $S(\mu_l(f)) = \mu_r(f)$, $S(\mu_r(f)) = \mu_l(f)$ and (2.6) holds for every $a \in X$ (or, more precisely, for each component of a with respect to the bigrading), then S is an antipode.

The proof will be given in Appendix 1. It is easy to check that $S^{D_\mathfrak{h}}$ is an antipode on $D_\mathfrak{h}$, where $\Delta^{D_\mathfrak{h}}$ is the canonical isomorphism and $\varepsilon^{D_\mathfrak{h}}$ the identity map. If one used the definition of [EV2], there would exist no antipode on $D_\mathfrak{h}$. Moreover, the last statement of the proposition would be false.

We now recall the generalized FRST construction. Let $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ be a finite-dimensional diagonalizable $\mathfrak{h}$-module and $R : \mathfrak{h}^* \to \mathrm{End}_\mathfrak{h}(V \otimes V)$ a meromorphic function. To each such R one associates an $\mathfrak{h}$-bialgebroid A_R. Though R is not a priori required to satisfy the QDYB equation, the construction is motivated by the case when it does.

Pick a homogeneous basis $\{e_x\}_{x \in X}$ of V, where X is an index set. Write R^{ab}_{xy} for the matrix elements

$$R(\lambda)(e_a \otimes e_b) = \sum_{xy} R^{ab}_{xy}(\lambda)\, e_x \otimes e_y$$

of R, and define $\omega : X \to \mathfrak{h}^*$ by $e_x \in V_{\omega(x)}$. The algebra A_R is generated by elements $\{L_{xy}\}_{x,y \in X}$, together with two copies of $M_{\mathfrak{h}^*}$, embedded as subalgebras. We will write the elements of these two copies as $f(\lambda)$, $f(\mu)$, respectively. The defining relations of A_R are

$$f(\lambda)L_{xy} = L_{xy}f(\lambda + \omega(x)), \quad f(\mu)L_{xy} = L_{xy}f(\mu + \omega(y)), \quad f(\lambda)g(\mu) = g(\mu)f(\lambda)$$

for f, $g \in M_{\mathfrak{h}^*}$, together with the RLL-relations

$$(2.10) \qquad \sum_{xy} R_{ac}^{xy}(\lambda) L_{xb} L_{yd} = \sum_{xy} R_{xy}^{bd}(\mu) L_{cy} L_{ax}.$$

The bigrading on A_R is defined by $L_{xy} \in A_{\omega(x), \omega(y)}$, $f(\lambda)$, $f(\mu) \in A_{00}$, and the moment maps by $\mu_l(f) = f(\lambda)$, $\mu_r(f) = f(\mu)$. Note that R must be $\mathfrak{h}$-invariant, or equivalently $R_{xy}^{ab} = 0$ for $\omega(x) + \omega(y) \neq \omega(a) + \omega(b)$, in order that the RLL-relations be consistent with the grading. Finally one defines a coproduct and a counit on A_R by

$$\Delta(L_{ab}) = \sum_{x \in X} L_{ax} \otimes L_{xb}, \qquad \Delta(f(\lambda)) = f(\lambda) \otimes 1, \qquad \Delta(f(\mu)) = 1 \otimes f(\mu),$$

$$\varepsilon(L_{ab}) = \delta_{ab} T_{-\omega(a)}, \qquad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f.$$

These definitions equip A_R with the structure of an $\mathfrak{h}$-bialgebroid.

2.2. **The** $\mathrm{SU}(2)$ **dynamical quantum group.** We will now write down in detail the results of the generalized FRST construction when applied to the dynamical R-matrix (1.1), where we think of q as a fixed number, $0 < q < 1$. We will denote the corresponding $\mathfrak{h}$-bialgebroid A_R by $\mathcal{F}_R(\mathrm{M}(2))$. It is a dynamical analogue of the algebra of polynomials on the space of complex 2×2-matrices. The four L-generators will be denoted by $\alpha = L_{11}$, $\beta = L_{1,-1}$, $\gamma = L_{-1,1}$, $\delta = L_{-1,-1}$. We also introduce the auxiliary functions

$$F(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1},$$

$$G(\lambda) = \frac{(q^{2(\lambda+1)} - q^2)(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2},$$

$$H(\lambda, \mu) = \frac{(q - q^{-1})(q^{2(\lambda+\mu+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)},$$

$$I(\lambda, \mu) = \frac{(q - q^{-1})(q^{2(\mu+1)} - q^{2(\lambda+1)})}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)}.$$

The following lemma is useful when checking various statements made below.

Lemma 2.3. *The functions F, G, H, I satisfy the following relations:*

$$q + q^{-1} = qF(\lambda) + \frac{q^{-1}}{F(\lambda - 1)}$$

$$H(\lambda, \mu) = qF(\lambda) - \frac{q^{-1}}{F(\mu - 1)} = qF(\mu) - \frac{q^{-1}}{F(\lambda - 1)},$$

$$I(\lambda, \mu) = q(F(\lambda) - F(\mu)) = q^{-1}\left(\frac{1}{F(\mu - 1)} - \frac{1}{F(\lambda - 1)}\right),$$

$$G(\lambda) = \frac{F(\lambda)}{F(\lambda - 1)},$$

$$G(\mu) - G(\lambda) = H(\lambda, \mu) I(\lambda, \mu).$$

We now write down the definition of $\mathcal{F}_R(\mathrm{M}(2))$. As in the non-dynamical case, the sixteen RLL-relations (2.10) reduce to six independent relations.

Definition 2.4. *The algebra $\mathcal{F}_R(\mathrm{M}(2))$ is generated by the four generators α, β, γ, δ, together with two copies of $M_{\mathfrak{h}^*}$, whose elements we write as $f(\lambda)$, $f(\mu)$. The defining relations are*

$$\alpha\beta = qF(\mu-1)\beta\alpha, \quad \alpha\gamma = qF(\lambda)\gamma\alpha, \quad \beta\delta = qF(\lambda)\delta\beta, \quad \gamma\delta = qF(\mu-1)\delta\gamma,$$

together with any two of the four relations

$$(2.11) \qquad \begin{aligned} \alpha\delta - \delta\alpha &= H(\lambda,\mu)\gamma\beta, & G(\mu)\alpha\delta - G(\lambda)\delta\alpha &= H(\lambda,\mu)\beta\gamma, \\ \beta\gamma - G(\mu)\gamma\beta &= I(\lambda,\mu)\delta\alpha, & \beta\gamma - G(\lambda)\gamma\beta &= I(\lambda,\mu)\alpha\delta, \end{aligned}$$

and, for arbitrary $f, g \in M_{\mathfrak{h}^*}$, $f(\lambda)g(\mu) = g(\mu)f(\lambda)$,

$$(2.12) \qquad \begin{aligned} f(\lambda)\alpha &= \alpha f(\lambda+1), & f(\mu)\alpha &= \alpha f(\mu+1), \\ f(\lambda)\beta &= \beta f(\lambda+1), & f(\mu)\beta &= \beta f(\mu-1), \\ f(\lambda)\gamma &= \gamma f(\lambda-1), & f(\mu)\gamma &= \gamma f(\mu+1), \\ f(\lambda)\delta &= \delta f(\lambda-1), & f(\mu)\delta &= \delta f(\mu-1). \end{aligned}$$

The bigrading $\mathcal{F}_R(\mathrm{M}(2)) = \bigoplus_{m,n\in\mathbb{Z},\, m+n\in 2\mathbb{Z}} \mathcal{F}_{mn}$ is defined on the generators by

$$\alpha \in \mathcal{F}_{11}, \quad \beta \in \mathcal{F}_{1,-1}, \quad \gamma \in \mathcal{F}_{-1,1}, \quad \delta \in \mathcal{F}_{-1,-1}, \quad f(\lambda),\, f(\mu) \in \mathcal{F}_{00}.$$

The coproduct $\Delta : \mathcal{F}_R(\mathrm{M}(2)) \to \mathcal{F}_R(\mathrm{M}(2)) \widetilde{\otimes} \mathcal{F}_R(\mathrm{M}(2))$ and counit $\varepsilon : \mathcal{F}_R(\mathrm{M}(2)) \to D_{\mathfrak{h}}$ are algebra homomorphisms defined on the generators by

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \gamma \otimes \beta + \delta \otimes \delta, \\ \Delta(f(\lambda)) &= f(\lambda) \otimes 1, & \Delta(f(\mu)) &= 1 \otimes f(\mu), \end{aligned}$$
$$\varepsilon(\alpha) = T_{-1}, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \quad \varepsilon(\delta) = T_1, \quad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f.$$

That any two of the relations (2.11) imply the others follows from the last identity of Lemma 2.3. By a straight-forward application of the diamond lemma [Be], any element in $\mathcal{F}_R(\mathrm{M}(2))$ can be written uniquely as a finite sum

$$\sum_{klmn} f_{klmn}(\lambda,\mu)\alpha^k\beta^l\gamma^m\delta^n,$$

where $f_{klmn} \in M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$ (and similarly for any other ordering of the generators).

Next we describe the dynamical analogue of the determinant.

Lemma 2.5. *The element*

$$\begin{aligned} c &= \frac{F(\lambda)}{F(\mu)}\delta\alpha - \frac{q^{-1}}{F(\mu)}\beta\gamma = \alpha\delta - qF(\lambda)\gamma\beta \\ &= \frac{F(\lambda-1)}{F(\mu-1)}\alpha\delta - qF(\lambda-1)\beta\gamma = \delta\alpha - \frac{q^{-1}}{F(\mu-1)}\gamma\beta \end{aligned}$$

is a central element of $\mathcal{F}_R(\mathrm{M}(2))$. Moreover, it satisfies $\Delta(c) = c \otimes c$, $\varepsilon(c) = 1$.

The proof is straightforward. That the four expressions are equal follows from Lemma 2.3 and (2.11). We write down the proof that $\alpha c = c\alpha$ in detail:

$$\alpha c = \alpha \left(\delta\alpha - \frac{q^{-1}}{F(\mu - 1)} \gamma\beta \right) = \alpha\delta\alpha - \frac{q^{-1}}{F(\mu - 2)} \alpha\gamma\beta$$

$$= \alpha\delta\alpha - \frac{F(\lambda)}{F(\mu - 2)} \gamma\alpha\beta = \alpha\delta\alpha - q\frac{F(\lambda)}{F(\mu - 2)} \gamma F(\mu - 1)\beta\alpha$$

$$= \alpha\delta\alpha - qF(\lambda)\gamma\beta\alpha = (\alpha\delta - qF(\lambda)\gamma\beta)\,\alpha = c\alpha.$$

The element c can also be obtained as a limit case of the elliptic determinant in [FV].

We can now introduce a dynamical analogue of the algebra of functions on the group $\mathrm{SL}(2, \mathbb{C})$.

Definition 2.6. *The algebra $\mathcal{F}_R(\mathrm{SL}(2))$ is the $\mathfrak{h}$-Hopf algebroid obtained by adjoining the relation $c = 1$ to $\mathcal{F}_R(\mathrm{M}(2))$ and defining the antipode by*

$$S(\alpha) = \frac{F(\lambda)}{F(\mu)}\,\delta, \quad S(\beta) = -\frac{q^{-1}}{F(\mu)}\,\beta, \quad S(\gamma) = -qF(\lambda)\gamma, \quad S(\delta) = \alpha,$$

$$S(f(\lambda)) = f(\mu), \quad S(f(\mu)) = f(\lambda).$$

We need to check that the antipode axioms are satisfied. By Proposition 2.2, it suffices to show that S reverses the defining relations of $\mathcal{F}_R(\mathrm{M}(2))$ and that (2.6) holds for the generators. This is a straight-forward verification. Note that, for the L-generators, (2.6) can be written compactly as

$$\begin{pmatrix} S(\alpha) & S(\beta) \\ S(\gamma) & S(\delta) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} S(\alpha) & S(\beta) \\ S(\gamma) & S(\delta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here the diagonal relations correspond to the four expressions for c given in Lemma 2.5, and the cross-diagonal relations are defining relations of $\mathcal{F}_R(\mathrm{M}(2))$.

Another application of the diamond lemma gives the first part of the following lemma. The second part is proved by a dimension count, cf. [KK] for the non-dynamical case.

Lemma 2.7. *The elements $\gamma^k\beta^l\alpha^m$, k, l, $m \geq 0$ and $\delta^k\gamma^l\beta^m$, $k > 0$, l, $m \geq 0$, form together a basis for $\mathcal{F}_R(\mathrm{SL}(2))$ as a module over $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*}) \simeq M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$. The elements $(\gamma^k\delta^l\alpha^m\beta^n)_{k+l+m+n=N}$ are for each N linearly independent over $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$.*

Next we will give a $*$-structure to our algebra. In general, to introduce a $*$-structure on an $\mathfrak{h}$-algebra A, we must assume that a conjugation (or, equivalently, a real form) $\lambda \mapsto \bar{\lambda}$ has been chosen on $\mathfrak{h}^*$. We can then define a $*$-structure on A to be a $\mathbb{C}$-antilinear and antimultiplicative involution $a \mapsto a^*$ on A such that $\mu_l(f)^* = \mu_l(\bar{f})$ and $\mu_r(f)^* = \mu_r(\bar{f})$, where $\bar{f}(\lambda) = \overline{f(\bar{\lambda})}$. It follows that $(A_{\alpha\beta})^* = A_{-\bar{\alpha}, -\bar{\beta}}$. A $*$-structure on an $\mathfrak{h}$-bialgebroid is in addition required to satisfy

$$(* \otimes *) \circ \Delta = \Delta \circ *, \qquad \varepsilon \circ * = *^{D_{\mathfrak{h}}} \circ \varepsilon,$$

where $*^{D_{\mathfrak{h}}}$ is defined by $(fT_\alpha)^* = T_{-\bar{\alpha}} \circ \bar{f} = (T_{-\bar{\alpha}}\bar{f})T_{-\bar{\alpha}}$.

Definition 2.8. *The algebra $\mathcal{F}_R(\mathrm{SU}(2))$ is the $\mathfrak{h}$-Hopf algebroid $\mathcal{F}_R(\mathrm{SL}(2))$ equipped with the $*$-structure $f(\lambda)^* = \bar{f}(\lambda)$, $f(\mu)^* = \bar{f}(\mu)$,*

$$\alpha^* = \delta, \quad \beta^* = -q\gamma, \quad \gamma^* = -q^{-1}\beta, \quad \delta^* = \alpha.$$

It is easy to check that the axioms for a $*$-structure are satisfied. Moreover, since S is invertible we may apply the following lemma. We indicate the proof in Appendix 1; in the case at hand it is easy to verify directly.

Lemma 2.9. *Let A be an $\mathfrak{h}$-Hopf algebroid equipped with a $*$-structure, such that the antipode S is invertible. Then S and $*$ are related by*

$$S \circ * \circ S \circ * = \mathrm{id}.$$

Remark 2.10. When $\lambda \to -\infty$, the R-matrix (1.1) tends to the R-matrix for the standard $\mathrm{SL}(2)$ quantum group. Accordingly, the Hopf algebra $\mathcal{F}_q(\mathrm{SL}(2))$ can be obtained as a formal limit of $\mathcal{F}_R(\mathrm{SL}(2))$ when $\lambda, \mu \to -\infty$. We will refer to this limit as the non-dynamical case. The formal limit of $\mathcal{F}_R(\mathrm{SL}(2))$ when $\lambda, \mu \to \infty$ is $\mathcal{F}_{q^{-1}}(\mathrm{SL}(2))$, which can also be viewed as $\mathcal{F}_q(\mathrm{SL}(2))$ with the opposite multiplication. We also need to consider the limit $\lambda \to -\infty$, $\mu \to \infty$, in which the defining relations of $\mathcal{F}_R(\mathrm{SL}(2))$ reduce to

$$\beta\alpha = q\alpha\beta, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \delta\gamma = q\gamma\delta,$$
$$\alpha\delta = \delta\alpha, \quad 1 = \alpha\delta - q\gamma\beta = q^2\alpha\delta - q\beta\gamma.$$

Replacing

$$(2.13) \qquad\qquad (\alpha, \beta, \gamma, \delta) \mapsto (\gamma, \alpha, -q^{-1}\delta, -q^{-1}\beta)$$

we again recover the algebra $\mathcal{F}_q(\mathrm{SL}(2))$. We also note that when $q \to 1$, the R-matrix (1.1) tends to the rational dynamical R-matrix

$$R'(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{\lambda+1} & 0 \\ 0 & \frac{1}{\lambda+1} & \frac{\lambda(\lambda+2)}{(\lambda+1)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding $\mathfrak{h}$-Hopf-algebroid $\mathcal{F}_{R'}(\mathrm{SL}(2))$ can be obtained as the formal limit of $\mathcal{F}_R(\mathrm{SL}(2))$ when $q \to 1$. This is a "quantum group without q" which has Racah polynomials (classical $6j$-symbols) as matrix elements for its corepresentations. Yet another interesting limit is the one giving rise to a Poisson–Lie groupoid, cf. [EV1].

2.3. **Some q-notation.** We will follow the standard notation of [GR], writing

$$(a;q)_k = \prod_{j=0}^{k-1}(1 - aq^j), \quad k \in \mathbb{Z}_{\geq 0},$$

$$(a_1, \ldots, a_n; q)_k = (a_1; q)_k \cdots (a_n; q)_k,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

$${}_{r+1}\phi_r \begin{bmatrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{bmatrix}; q, z \end{bmatrix} = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k,$$

$${}_{r+1}W_r(a; b_1, \ldots, b_{r-2}; q, z) = {}_{r+1}\phi_r \begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, \ldots, b_{r-2} \\ \sqrt{a}, -\sqrt{a}, aq/b_1, \ldots, aq/b_{r-2} \end{bmatrix}; q, z \end{bmatrix}$$

$$= \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(a, b_1, \ldots, b_{r-2}; q)_k}{(q, aq/b_1, \ldots, aq/b_{r-2}; q)_k} z^k.$$

Occasionally we will write

$$(2.14) \qquad\qquad (a; q)_x = \prod_{j=0}^{\infty} \frac{1 - aq^j}{1 - aq^{x+j}}, \quad x \in \mathbb{C},$$

so that

$$(2.15) \qquad\qquad (a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n}.$$

We will write

$$(2.16) \qquad h_k(\cos\theta, a; q) = (ae^{i\theta}, ae^{-i\theta}; q)_k = \prod_{j=0}^{k-1} \left(1 - 2aq^j \cos\theta + a^2 q^{2j}\right);$$

this is a polynomial of degree k in $\cos\theta$ which may be called the Askey–Wilson monomial.

We will mainly encounter terminating ${}_4\phi_3$- and ${}_8W_7$-series, for which we recall the transformation formulas

$${}_8W_7(a; q^{-n}, b, c, d, e; q, z)$$

$$(2.17) \quad = \frac{(aq, aq/bc, aq/bd, aq/be; q)_n}{(aq/b, aq/c, aq/d, aq/e; q)_n} b^n \, {}_4\phi_3 \begin{bmatrix} q^{-n}, b, bq^{-n}/a, q/z \\ bcq^{-n}/a, bdq^{-n}/a, beq^{-n}/a \end{bmatrix}; q, q \end{bmatrix},$$

$$(2.18) \quad = \frac{(aq, b, q/z; q)_n}{(aq/c, aq/d, aq/e; q)_n} \left(-q^{-\frac{1}{2}(n+1)} z\right)^n \, {}_4\phi_3 \begin{bmatrix} q^{-n}, aq/bc, aq/bd, aq/be \\ q^{1-n}/b, aq/b, q^{-n}z \end{bmatrix}; q, q \end{bmatrix},$$

where $n \in \mathbb{Z}_{\geq 0}$ and $z = a^2 q^{n+2}/bcde$. They are obtained by combining equations (III.15) and (III.18) in [GR]; note also that the ${}_4\phi_3$'s are obtained from each other by inverting the order of summation.

Finally we recall the q-Racah and Askey–Wilson polynomials, both introduced by Askey and Wilson [AW1, AW2], cf. also [GR]. The q-Racah polynomials are defined

by

$$(2.19) \qquad R_n(\mu(x); a, b, c, d; q) = {}_4\phi_3 \left(\begin{array}{c} q^{-n}, abq^{n+1}, q^{-x}, cdq^{x+1} \\ aq, \ bdq, \ cq \end{array} ; q, q \right),$$

where it is assumed that one of the quantities aq, bdq or cq equals q^{-N} with $N \in \mathbb{Z}_{\geq 0}$, $n \leq N$. This is a polynomial of degree n in $\mu(x) = q^{-x} + cdq^{x+1}$. For generic values of the parameters, $\{R_n\}_{n=0}^N$ is a system of orthogonal polynomials, with the orthogonality measure supported on $\mu(\{0, 1, \dots, N\})$. For later use we recall the symmetries

$$(2.20) \qquad R_n(\mu(x); a, b, c, d; q) = R_x(\mu(n); a, dc/a, c, ba/c; q),$$

which is obvious from the definition, and

$$
\begin{aligned}
R_n(\mu(x); a, b, q^{-N-1}, d; q) &= d^n \frac{(bq, aq/d; q)_n}{(aq, bdq; q)_n} \\
(2.21) \qquad\qquad & \times R_{N-x}(\mu(n); b, q^{-N-1}/bd, q^{-N-1}, q^{N+1}ab; q) \\
&= (dq^{x-N})^x \frac{(aq^{1+N-x}/d, bq^{1+N-x}; q)_x}{(aq, bdq; q)_x} \\
(2.22) \qquad\qquad & \times R_{N-n}(\mu(x); dq^{-N-1}/a, q^{-N-1}/bd, q^{-N-1}, d; q),
\end{aligned}
$$

which follows from [GR, (III.15)]. The Askey–Wilson polynomials are defined by

$$(2.23) \ \ p_n(\cos\theta; a, b, c, d; q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3 \left(\begin{array}{c} q^{-n}, abcdq^{n+1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} ; q, q \right);$$

this is a polynomial of degree n in $\cos\theta$ which is symmetric in the four parameters a, b, c, d.

3. Corepresentations

3.1. **Corepresentations of $\mathfrak{h}$-bialgebroids.** In this section we will discuss corepresentations of $\mathfrak{h}$-bialgebroids (or better $\mathfrak{h}$-coalgebroids, cf. §5.1). As in the case of representations (cf. §4.1), it is natural to view corepresentation spaces as "dynamical" spaces. In [EV2] a category of so called $\mathfrak{h}$-vector spaces is introduced, whose objects are complex vector spaces but whose morphisms are $\mathbb{C}$-linear maps $V \to M_{\mathfrak{h}^*} \otimes W$. We choose to work instead with vector spaces over $M_{\mathfrak{h}^*}$, with $M_{\mathfrak{h}^*}$-linear maps as morphisms. We must point out, however, that this is purely a matter of taste, and that we could equivalently have used the category introduced in [EV2].

Thus we define an $\mathfrak{h}$-*space* to be a vector space over $M_{\mathfrak{h}^*}$, which is also a diagonalizable $\mathfrak{h}$-module, $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, with $M_{\mathfrak{h}^*} V_\alpha \subseteq V_\alpha$ for all α. A morphism of $\mathfrak{h}$-spaces is an $\mathfrak{h}$-invariant (that is, grade-preserving) $M_{\mathfrak{h}^*}$-linear map.

If A is an $\mathfrak{h}$-algebra and V an $\mathfrak{h}$-space, we define $A \widetilde{\otimes} V = \bigoplus_{\alpha\beta} A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations

$$(3.1) \qquad\qquad \mu_r^A(f) a \otimes v = a \otimes fv.$$

The grading $A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta \subseteq (A \widetilde{\otimes} V)_\alpha$ and the extension of scalars $f(a \otimes v) = \mu_l^A(f) a \otimes v$ make $A \widetilde{\otimes} V$ into an $\mathfrak{h}$-space. This definition is compatible with the matrix tensor

product of $\mathfrak{h}$-algebras in the sense that $(A\widetilde{\otimes}B)\widetilde{\otimes}V = A\widetilde{\otimes}(B\widetilde{\otimes}V)$ when A and B are $\mathfrak{h}$-algebras and V an $\mathfrak{h}$-space.

We can now define a (left) *corepresentation* of an $\mathfrak{h}$-bialgebroid A on an $\mathfrak{h}$-space V to be an $\mathfrak{h}$-space morphism $\pi : V \to A\widetilde{\otimes}V$ such that

$$(3.2) \qquad (\Delta \otimes \mathrm{id}) \circ \pi = (\mathrm{id} \otimes \pi) \circ \pi, \qquad (\varepsilon \otimes \mathrm{id}) \circ \pi = \mathrm{id}.$$

The first of these equalities is in the sense of the natural isomorphism $(A\widetilde{\otimes}A)\widetilde{\otimes}V \simeq A\widetilde{\otimes}(A\widetilde{\otimes}V)$, the second one in terms of the isomorphism $D_{\mathfrak{h}}\widetilde{\otimes}V \simeq V$ defined by $f\,T_{-\alpha} \otimes v \simeq fv$, $f \in M_{\mathfrak{h}^*}$, $v \in V_\alpha$. A morphism or *intertwiner* of corepresentations is an $\mathfrak{h}$-space morphism $\phi : V_1 \to V_2$ such that

$$(3.3) \qquad \pi_2 \circ \phi = (\mathrm{id} \otimes \phi) \circ \pi_1,$$

where $\pi_i : V_i \to A\widetilde{\otimes}V_i$ are corepresentations (note that $\mathrm{id} \otimes \phi$ factors to a map on $A\widetilde{\otimes}V_1$).

If we pick a homogeneous basis $\{v_k\}_k$ of V (over $M_{\mathfrak{h}^*}$), $v_k \in V_{\omega(k)}$, and introduce the matrix elements $t_{kj} \in A$ by

$$(3.4) \qquad \pi(v_k) = \sum_j t_{kj} \otimes v_j,$$

which is possible in view of (3.1), then (3.2) may be stated as

$$(3.5) \qquad \Delta(t_{kl}) = \sum_j t_{kj} \otimes t_{jl}, \qquad \varepsilon(t_{kl}) = \delta_{kl}\,T_{-\omega(k)}.$$

This refers only to the complex vector space spanned by the chosen basis. Thus, as long as we consider a single corepresentation, the dynamical variables play no role. However, if V_1 and V_2 are two corepresentations with matrix elements t_{kj}^1, t_{kj}^2, with respect to some bases $\{v_k^1\}_k$, $\{v_k^2\}_k$, and $\phi : V_1 \to V_2$ an intertwiner with matrix elements $\phi_{kj} \in M_{\mathfrak{h}^*}$ defined by

$$(3.6) \qquad \phi(v_k^1) = \sum_j \phi_{kj}v_j^2,$$

then (3.3) may be written as

$$(3.7) \qquad \sum_j \phi_{jl}(\mu)t_{kj}^1 = \sum_j \phi_{kj}(\lambda)t_{jl}^2 \quad \text{for all } k,\, l,$$

so the dynamical variables appear when considering intertwiners. For later use we observe that if A is an $\mathfrak{h}$-Hopf algebroid, then it follows from (2.6) and (3.5) that

$$(3.8) \qquad \delta_{kl} = \sum_j S(t_{kj})t_{jl} = \sum_j t_{kj}S(t_{jl}).$$

If A is an $\mathfrak{h}$-bialgebroid, viewed as an $\mathfrak{h}$-space with $A_\alpha = \bigoplus_\beta A_{\alpha\beta}$, $fv = \mu_l(f)v$, then the coproduct Δ defines a corepresentation of A on itself, the *regular corepresentation*. More generally, if V is a subspace of A with $\Delta(V) \subseteq A\widetilde{\otimes}V$ and $\mu_l(M_{\mathfrak{h}^*})V \subseteq V$, then $\Delta\big|_V$ defines a corepresentation of A on V.

3.2. **Corepresentations of $\mathcal{F}_R(\mathrm{SL}(2))$.** Let V_N be the subspace of $\mathcal{F}_R(\mathrm{M}(2))$ spanned by $\{\gamma^{N-k}\alpha^k\}_{k=0}^N$ together with $\mu_l(M_{\mathfrak{h}^*})$. It is easy to see, and will be made clear below, that $\Delta(V_N) \subseteq \mathcal{F}_R(\mathrm{M}(2)) \widetilde{\otimes} V_N$, so that $\Delta\big|_{V_N}$ is a corepresentation. In this section we will compute the matrix elements of these corepresentations. Our method follows that of [K1] for the non-dynamical case. The following lemma will be used repeatedly.

Lemma 3.1. *The following relations hold in the algebra $\mathcal{F}_R(\mathrm{M}(2))$:*

$$\alpha^n \beta^m = q^{mn} \frac{(q^{-2(\mu+m)}; q^2)_n}{(q^{-2\mu}; q^2)_n} \beta^m \alpha^n,$$

$$\alpha^n \gamma^m = q^{mn} \frac{(q^{-2(\lambda+m+1)}; q^2)_n}{(q^{-2(\lambda+1)}; q^2)_n} \gamma^m \alpha^n,$$

$$\beta^n \delta^m = q^{mn} \frac{(q^{-2(\lambda+m+1)}; q^2)_n}{(q^{-2(\lambda+1)}; q^2)_n} \delta^m \beta^n,$$

$$\gamma^n \delta^m = q^{mn} \frac{(q^{-2(\mu+m)}; q^2)_n}{(q^{-2\mu}; q^2)_n} \delta^m \gamma^n.$$

Proof. We prove the second relation. The other ones are derived similarly or by observing that the four subalgebras generated by $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$, $\{\beta, \delta\}$ and $\{\gamma, \delta\}$ are all isomorphic. It is clear from the defining relations that

$$\alpha^n \gamma^m = C_{mn}(\lambda) \gamma^m \alpha^n$$

for some $C_{mn} \in M_{\mathfrak{h}^*}$. Multiplying with α from the left gives

$$\alpha^{n+1}\gamma^m = \alpha\, C_{mn}(\lambda)\gamma^m\alpha^n = C_{mn}(\lambda-1)\alpha\gamma^m\alpha^n = C_{mn}(\lambda-1)C_{m1}(\lambda)\gamma^m\alpha^{n+1},$$

leading to the recursion relation

$$C_{m,n+1}(\lambda) = C_{mn}(\lambda-1)C_{m1}(\lambda),$$

while multiplying with γ from the right similarly leads to

$$C_{m+1,n}(\lambda) = C_{mn}(\lambda)C_{1n}(\lambda+m).$$

The coefficients C_{mn} are determined by these two recursion relations together with the initial conditions $C_{m0} = C_{0n} = 1$, $C_{11} = qF$. It is easy to check that the solution is indeed given by $C_{mn}(\lambda) = q^{mn}(q^{-2(\lambda+m+1)}; q^2)_n/(q^{-2(\lambda+1)}; q^2)_n$. $\qquad\square$

We can now compute the matrix elements of our corepresentations. In what follows it will be convenient to write

$$(3.9) \qquad f(\lambda) = f(\lambda) \otimes 1, \qquad f(\rho) = f(\mu) \otimes 1 = 1 \otimes f(\lambda), \qquad f(\mu) = 1 \otimes f(\mu)$$

for the three dynamical variables present in a tensor product $A \widetilde{\otimes} A$.

Proposition 3.2. *In the algebra $\mathcal{F}_R(\mathrm{M}(2))$,*

$$(3.10) \qquad \Delta(\gamma^{N-k}\alpha^k) = \sum_{j=0}^N t_{kj}^N \otimes \gamma^{N-j}\alpha^j,$$

where the matrix elements t_{kj}^N are given by

$$t_{kj}^N = \sum_{l=\max(0,\,j+k-N)}^{\min(j,k)} \begin{bmatrix} N-k \\ j-l \end{bmatrix}_{q^2} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{j(j+2k-N)+l(3l-3k-3j+N)}$$

$$\times \frac{(q^{2(j-N-\mu-1)};q^2)_{j-l}}{(q^{2(j+k-l-N-\mu-1)};q^2)_{j-l}} \, \gamma^{j-l} \delta^{N-k-j+l} \alpha^l \beta^{k-l}.$$

Note that $\gamma^{N-k}\alpha^k \in \mathcal{F}_R(\mathrm{M}(2))_{2k-N,N}$, which implies that $t_{kj}^N \in \mathcal{F}_R(\mathrm{M}(2))_{2k-N,2j-N}$. This is of course in agreement with the proposition.

Proof. We will first prove the case $k = N$, which may be written as

$$(3.11) \qquad \Delta(\alpha^k) = \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{l(l-k)} \alpha^l \beta^{k-l} \otimes \gamma^{k-l}\alpha^l.$$

It is clear from the defining relations that

$$\Delta(\alpha^k) = (\alpha \otimes \alpha + \beta \otimes \gamma)^k = \sum_{l=0}^{k} C_{kl}(\rho) \alpha^l \beta^{k-l} \otimes \gamma^{k-l}\alpha^l$$

for some coefficients $C_{kl} \in M_{\mathfrak{h}^*}$. To find a recursion formula for C_{kl} we write

$$\Delta(\alpha^{k+1}) = (\alpha \otimes \alpha + \beta \otimes \gamma) \sum_{l} C_{kl}(\rho) \alpha^l \beta^{k-l} \otimes \gamma^{k-l}\alpha^l$$

$$= \sum_{l} C_{kl}(\rho-1) \alpha^{l+1} \beta^{k-l} \otimes \alpha\gamma^{k-l}\alpha^l + C_{kl}(\rho+1) \beta\alpha^l \beta^{k-l} \otimes \gamma^{k-l+1}\alpha^l$$

$$= \sum_{l} C_{kl}(\rho-1) q^{k-l} \frac{1 - q^{-2(\rho+k-l+1)}}{1 - q^{-2(\rho+1)}} \, \alpha^{l+1} \beta^{k-l} \otimes \gamma^{k-l}\alpha^{l+1}$$

$$+ C_{kl}(\rho+1) q^{-l} \frac{1 - q^{-2(\rho-l+1)}}{1 - q^{-2(\rho+1)}} \, \alpha^l \beta^{k-l+1} \otimes \gamma^{k-l+1}\alpha^l,$$

where we used Lemma 3.1 in the last step. This leads to the recursion

$$C_{k+1,l}(\rho) = C_{k,l-1}(\rho-1) q^{k-l+1} \frac{1 - q^{-2(\rho+k-l)}}{1 - q^{-2(\rho+1)}} + C_{kl}(\rho+1) q^{-l} \frac{1 - q^{-2(\rho-l+1)}}{1 - q^{-2(\rho+1)}}$$

for $l = 0, \ldots, k+1$, where $C_{k,-1} = C_{k,k+1} = 0$. We must check that this holds for the constant functions $C_{kl}(\rho) = \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{l(l-k)}$. After simplifications one arrives at

$$(3.12) \qquad \left(1 - q^{-2(\rho+1)}\right) \begin{bmatrix} k+1 \\ l \end{bmatrix}_{q^2} = \left(q^{2(k-l+1)} - q^{-2(\rho+1)}\right) \begin{bmatrix} k \\ l-1 \end{bmatrix}_{q^2}$$

$$+ \left(1 - q^{2l} q^{-2(\rho+1)}\right) \begin{bmatrix} k \\ l \end{bmatrix}_{q^2},$$

which is equivalent to the two different Pascal's triangle identities for the q-binomial coefficients [GR]:

$$\begin{bmatrix} k+1 \\ l \end{bmatrix}_{q^2} = q^{2(k-l+1)} \begin{bmatrix} k \\ l-1 \end{bmatrix}_{q^2} + \begin{bmatrix} k \\ l \end{bmatrix}_{q^2},$$

$$\begin{bmatrix} k+1 \\ l \end{bmatrix}_{q^2} = \begin{bmatrix} k \\ l-1 \end{bmatrix}_{q^2} + q^{2l} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2}.$$

The case $k = 0$ of the proposition, which may be written as

$$\Delta(\gamma^{N-k}) = \sum_{m=0}^{N-k} \begin{bmatrix} N-k \\ m \end{bmatrix}_{q^2} q^{m(m-N+k)} \gamma^m \delta^{N-k-m} \otimes \gamma^{N-k-m} \alpha^m,$$

may be proved in the same way. Multiplying this expression and (3.11) gives

$$\begin{aligned}
\Delta(\gamma^{N-k}\alpha^k) &= \sum_{m=0}^{N-k}\sum_{l=0}^{k} \begin{bmatrix} N-k \\ m \end{bmatrix}_{q^2} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{m(m-N+k)+l(l-k)} \\
&\quad \times \gamma^m \delta^{N-k-m}\alpha^l \beta^{k-l} \otimes \gamma^{N-k-m}\alpha^m\gamma^{k-l}\alpha^l \\
&= \sum_{m=0}^{N-k}\sum_{l=0}^{k} \begin{bmatrix} N-k \\ m \end{bmatrix}_{q^2} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{m(m-N+k)+l(l-k)+m(k-l)} \\
&\quad \times \frac{(q^{-2(\rho+N-l-m+1)};q^2)_m}{(q^{-2(\rho+N-k-m+1)};q^2)_m} \gamma^m \delta^{N-k-m}\alpha^l\beta^{k-l} \otimes \gamma^{N-l-m}\alpha^{l+m}
\end{aligned}$$

by Lemma 3.1. Putting $m = j - l$ and simplifying completes the proof. $\qquad\square$

We will now consider V_N as a corepresentation space of the quotient algebra $\mathcal{F}_R(\mathrm{SL}(2))$ of $\mathcal{F}_R(\mathrm{M}(2))$. Using the determinant relation we will factor each matrix element as a trivial part times a function involving only commuting variables. The following lemma is the key to finding these factorizations.

Lemma 3.3. *The element Ξ defined by*

$$
(3.13) \qquad \begin{aligned}
\Xi &= q^{\lambda-\mu+1} + q^{\mu-\lambda-1} - q^{-(\lambda+\mu+2)}(1-q^{2(\lambda+2)})(1-q^{2\mu})\gamma\beta \\
&= q^{\lambda-\mu-1} + q^{\mu-\lambda+1} - q^{-(\lambda+\mu+2)}(1-q^{2(\lambda+1)})(1-q^{2(\mu+1)})\beta\gamma
\end{aligned}
$$

is a central element of $\mathcal{F}_R(\mathrm{SL}(2))$. Moreover, it satisfies $\varepsilon(\Xi) = 0$, $S(\Xi) = \Xi$ and $\Xi^* = \Xi$ with respect to the $\mathcal{F}_R(\mathrm{SU}(2))$ $*$-structure.

Note that the algebra $\mathcal{F}_q(\mathrm{SL}(2))$ has trivial center. Therefore, the existence of Ξ is a purely dynamical phenomenon. In §6.1 we will see that Ξ plays the role of Casimir operator in the representation theory of $\mathcal{F}_R(\mathrm{SU}(2))$.

The proof of Lemma 3.3 is straight-forward. That the two expressions for Ξ are equal follows from the determinant relation. To prove centrality, we first check that $\Xi^* = \Xi$. Then it is enough to prove that Ξ commutes with α, β, and $f(\lambda)$, $f(\mu)$ for $f \in M_{\mathfrak{h}^*}$. To check that Ξ commutes with β one should write $\beta\Xi$ using the first and $\Xi\beta$ using the second expression in (3.13).

We will need the following relations in the subalgebra $\mathcal{F}_R(\mathrm{SL}(2))_{00}$ of $\mathcal{F}_R(\mathrm{SL}(2))$. This is a commutative algebra generated by 1, Ξ, $\mu_l(M_{\mathfrak{h}^*})$ and $\mu_r(M_{\mathfrak{h}^*})$.

Lemma 3.4. *In the algebra* $\mathcal{F}_R(\mathrm{SL}(2))_{00}$, *the following relations hold:*

$$\alpha^k \delta^k = \frac{1}{(q^{-2(\lambda+1)}, q^{-2\mu}; q^2)_k} \, h_k(\tfrac{1}{2}\,\Xi, q^{-(\lambda+\mu+1)}; q^2),$$

$$\delta^k \alpha^k = \frac{1}{(q^{2(\lambda+2)}, q^{2(\mu+1)}; q^2)_k} \, h_k(\tfrac{1}{2}\,\Xi, q^{\lambda+\mu+3}; q^2),$$

$$\beta^k \gamma^k = \frac{(-q^{-1})^k}{(q^{-2(\lambda+1)}, q^{2(\mu+1)}; q^2)_k} \, h_k(\tfrac{1}{2}\,\Xi, q^{\mu-\lambda+1}; q^2),$$

$$\gamma^k \beta^k = \frac{(-q)^k}{(q^{2(\lambda+2)}, q^{-2\mu}; q^2)_k} \, h_k(\tfrac{1}{2}\,\Xi; q^{\lambda-\mu+1}; q^2),$$

where we use the notation (2.16).

To prove Lemma 3.4, one checks that the case $k = 1$ follows from the determinant relation and the definition of Ξ. The general case then follows immediately by induction on k, using that Ξ is central.

We can now prove the main result of this section.

Theorem 3.5. *In the algebra* $\mathcal{F}_R(\mathrm{SL}(2))$, *the matrix elements* t_{kj}^N *are given by*

$$t_{kj}^N = \begin{cases} \gamma^{j-k}\delta^{N-k-j}P_k, & k \le j, \ k+j \le N, \\ \gamma^{j-k}P_{N-j}\,\alpha^{k+j-N}, & k \le j, \ N \le k+j, \\ \delta^{N-k-j}P_j\,\beta^{k-j}, & j \le k, \ k+j \le N, \\ P_{N-k}\,\alpha^{k+j-N}\beta^{k-j}, & j \le k, \ N \le k+j, \end{cases}$$

where $P_n \in \mathcal{F}_R(\mathrm{SL}(2))_{00}$ *can be written in terms of Askey–Wilson polynomials, cf.* (2.23), *as*

$$P_n = q^{n(\lambda-\mu+1+n+|N-k-j|)+j(j-N)}\frac{(q^2; q^2)_{N-n}}{(q^2; q^2)_j(q^2; q^2)_{N-j}(q^{2(\lambda+2)}, q^{2(-\mu+|N-k-j|)}; q^2)_n}$$
$$\times\, p_n(\tfrac{1}{2}\,\Xi; q^{\mu-\lambda+1+2|j-k|}, q^{\lambda-\mu+1}, q^{\lambda+\mu+3}, q^{-\lambda-\mu-1+2|N-k-j|}; q^2).$$

In §7 we will see that the Schur-type orthogonality relations for matrix elements correspond to the orthogonality of Askey–Wilson polynomials. We also remark that in the alternative notation of [NM],

(3.14) $\qquad p_n^{(\alpha,\beta)}(x; s, t|q) = p_n(x; q^{\frac{1}{2}}t/s, q^{\frac{1}{2}+\alpha}s/t, -q^{\frac{1}{2}}/st, -q^{\frac{1}{2}+\beta}st; q),$

the Askey–Wilson polynomial of Theorem 3.5 may be suggestively written as

$$p_n^{(|j-k|,|N-k-j|)}(\tfrac{1}{2}\,\Xi; iq^{-\lambda-1}, iq^{-\mu-1}; q^2).$$

After the preparations that we have made, the proof of Theorem 3.5 is straightforward. For instance, in the case $k \le j$, $k+j \le N$, Proposition 3.2 gives

$$t_{kj}^N = \sum_{l=0}^{k} C_l(\mu)\gamma^{j-l}\delta^{N-k-j+l}\alpha^l\beta^{k-l},$$

where

$$C_l(\mu) = \begin{bmatrix} N-k \\ j-l \end{bmatrix}_{q^2} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{j(j+2k-N)+l(3l-3k-3j+N)} \frac{(q^{2(j-N-\mu-1)};q^2)_{j-l}}{(q^{2(j+k-l-N-\mu-1)};q^2)_{j-l}}.$$

Using Lemma 3.1 repeatedly one can rewrite this expression as

$$t_{kj}^N = \gamma^{j-k}\delta^{N-k-j} \sum_{l=0}^{k} D_l(\lambda,\mu)\gamma^{k-l}\beta^{k-l}\delta^l\alpha^l,$$

where one computes

$$D_l(\lambda,\mu) = \begin{bmatrix} N-k \\ j-l \end{bmatrix}_{q^2} \begin{bmatrix} k \\ l \end{bmatrix}_{q^2} q^{j(j-N)+k(N-k)+jk+l(3l-2k-2j)}$$

$$\times \frac{(q^{2(-\mu-j-1)};q^2)_{j-l}(q^{-2\mu};q^2)_{k-l}(q^{2(-\mu-l)};q^2)_l(q^{2(-\lambda+l-k-1)};q^2)_{k-l}}{(q^{2(-\mu+k-l-j-1)};q^2)_{j-l}(q^{2(-\mu+N-k-j)};q^2)_{k-l}(q^{2(-\mu+k-2l)};q^2)_l(q^{2(-\lambda-1-k)};q^2)_{k-l}}.$$

Plugging in the expressions from Lemma 3.4 and using elementary identities for q-shifted factorials gives an expression like the one we are looking for, but with

$$P_k = \begin{bmatrix} N-k \\ j \end{bmatrix}_{q^2} q^{k(2\mu+2+3j-N)+j(j-N)} \frac{(q^{-2(\mu+1+j)},q^{\lambda-\mu+1}\xi,q^{\lambda-\mu+1}\xi^{-1};q^2)_k}{(q^{-2(\mu+1)},q^{2(\lambda+2)},q^{2(\mu+1-N+j)};q^2)_k}$$

$$\times {}_8W_7\left(q^{2(\mu+1-k)};q^{-2k},q^{-2j},q^{2(\mu+1-N+j)},q^{\lambda+\mu+3}\xi,q^{\lambda+\mu+3}\xi^{-1};q^2,q^{2(N-k-\lambda)}\right),$$

where $\xi + \xi^{-1} = \Xi$ (here ξ is a formal quantity used to write the above expression in standard q-notation). Applying (2.18) gives the desired expression for P_k. In the remaining three cases, the theorem can be proved similarly, or derived using symmetries of the matrix elements, cf. Remark 3.15 below.

We conclude this section by showing that analogues of the Peter–Weyl theorem and Schur's Lemma hold for $\mathcal{F}_R(\mathrm{SL}(2))$.

Proposition 3.6. *The matrix elements $\{t_{kj}^N\}$ for k, j, $N \in \mathbb{Z}_{\geq 0}$, $k, j \leq N$ form a basis for $\mathcal{F}_R(\mathrm{SL}(2))$ as a module over $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$.*

Proof. Let I denote the set of invertible elements in $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$. First we observe that the element P_n of Theorem 3.5 is a polynomial in Ξ of degree n with the leading coefficient in I. By Lemma 3.4, $\gamma^k\beta^k$ is a polynomial in Ξ of degree k, with the leading coefficient in I. Applying Gauss elimination, we can expand $\Xi^k = \sum_l c_l \gamma^l \beta^l$, again with the leading coefficient $c_k \in I$. Combining these facts and using Lemma 3.1, we can for $k+j \leq N$ write $t_{kj}^N = \sum_{l=0}^{\min(j,k)} d_l \delta^{N-k-j}\gamma^{j-l}\beta^{k-l}$ with $d_0 \in I$. Again by Gauss elimination, each element $\delta^l\gamma^m\beta^n$ is in the $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$-span of $\{t_{kj}^N\}_{k+j\leq N}$. Similarly, $\gamma^l\beta^m\alpha^n$ is in the $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$-span of $\{t_{kj}^N\}_{k+j\geq N}$. Thus, by Lemma 2.7, the matrix elements span $\mathcal{F}_R(\mathrm{SL}(2))$. A dimension count completes the proof. $\square$

Remark 3.7. It is easy to check that, for any $0 \neq g \in M_{\mathfrak{h}^*}$ and $0 \leq j \leq N$, the coproduct restricted to $\bigoplus_{k=0}^N \mu_r(g)\mu_l(M_{\mathfrak{h}^*})t_{kj}^N$ is a corepresentation equivalent

to V_N, $\gamma^{N-k}\alpha^k \mapsto g(\mu)t_{kj}^N$ being an intertwiner. Then Proposition 3.6 gives the decomposition

$$\mathcal{F}_R(\mathrm{SL}(2)) \simeq \bigoplus_{N=0}^{\infty} V_N \otimes M_{\mathfrak{h}^*}^{N+1}$$

of the regular corepresentation. In contrast to the non-dynamical case, each V_N occurs with infinite multiplicity.

Corollary 3.8. *The corepresentations V_N are irreducible, that is, if $U \subseteq V_N$ is an $\mathfrak{h}$-subspace which is invariant in the sense that $\Delta(U) \subseteq A\widetilde{\otimes}U$, then $U = 0$ or $U = V_N$.*

Proof. That U is an $\mathfrak{h}$-subspace means precisely that $U = \sum_{k\in\Lambda} \mu_l(M_{\mathfrak{h}^*})\gamma^{N-k}\alpha^k$ for some $\Lambda \subseteq \{0, \ldots, N\}$. Then the invariance of U means that $t_{kj}^N = 0$ for $k \in \Lambda$, $j \notin \Lambda$. By Proposition 3.6, this is impossible unless $U = 0$ or $U = V_N$. $\qquad\square$

Corollary 3.9. *If $\phi : V_M \to V_N$ is an intertwining map, then $\phi = 0$ for $M \neq N$ and $\phi = Z\,\mathrm{id}$ for some $Z \in \mathbb{C}$ otherwise.*

Proof. First one checks that the kernel and the image of an intertwiner are always invariant $\mathfrak{h}$-subspaces. It follows that an intertwiner between irreducible corepresentations is either zero or bijective. Counting dimensions (over $M_{\mathfrak{h}^*}$) then gives the first statement.

For the second statement, we assume that $\phi : V_M \to V_M$. Since ϕ preserves the grading we have $\phi(\gamma^{M-k}\alpha^k) = \phi_k\gamma^{M-k}\alpha^k$ with $\phi_k \in M_{\mathfrak{h}^*}$. By (3.7), the intertwining property means that $\phi_l(\mu)t_{kl}^M = \phi_k(\lambda)t_{kl}^M$ for all k, l. By Proposition 3.6, this implies that $\phi_k(\lambda)$ is independent of k and λ, which completes the proof. $\qquad\square$

More generally, if V is a corepresentation of an $\mathfrak{h}$-bialgebroid such that its matrix elements are linearly independent over $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$, then it follows from (3.7) that any intertwiner $V \to V$ is a complex multiple of the identity. We expect this to be true for all irreducible corepresentations of a large class of interesting dynamical quantum groups. The following example shows that it is not true for general $\mathfrak{h}$-Hopf algebroids. This suggests that one might adopt the Peter–Weyl theorem as an axiom for dynamical quantum groups, which would give an approach similar to that of [DK] in the non-dynamical case.

Example 3.10. This example is modelled on the group of rotations of the plane, to which it formally reduces when $\lambda = -1$. Let $A = M_{\mathfrak{h}^*}[x, y]$ be the algebra of polynomials in two commuting variables over the field $M_{\mathfrak{h}^*}$, $\mathfrak{h}^* = \mathbb{C}$. Define the coproduct and counit by

$$\Delta(x) = x \otimes x + \lambda y \otimes y, \quad \Delta(y) = y \otimes x + x \otimes y, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.$$

The moment maps $\mu_l(f) = \mu_r(f) = f$ and the bigrading $A = A_{00}$ give A the structure of an $\mathfrak{h}$-bialgebroid. Let V be the two-dimensional subspace consisting of homogeneous polynomials of degree 1. Then $\Delta|_V$ is a corepresentation of A. One easily checks that if $fx + gy$ spans a one-dimensional invariant subspace, then

$g(\lambda)^2 = \lambda f(\lambda)^2$, which has no meromorphic solutions. Therefore, V is irreducible. On the other hand, the intertwiners from V to V are given by

$$C(x) = fx + \lambda gy, \qquad C(y) = gx + fy$$

for f, $g \in M_{\mathfrak{h}^*}$ arbitrary. To get a similar example for $\mathfrak{h}$-Hopf algebroids, adjoin the relation $x^2 - \lambda y^2 = 1$ and define the antipode by $S(x) = x$, $S(y) = -y$.

3.3. **Unitarity of the corepresentations.** Our next task is to show that our corepresentations are, in a certain sense, unitarizable.

Definition 3.11. *A corepresentation of a $*$-$\mathfrak{h}$-Hopf algebroid A on an $\mathfrak{h}$-space V is unitarizable if there exists a basis of V such that the corresponding matrix elements, cf. (3.4), satisfy*

$$\Gamma_k(\mu)S(t_{kj})^* = \Gamma_j(\lambda)t_{jk}$$

for some $0 \neq \Gamma_k \in M_{\mathfrak{h}^*}$ with $\bar{\Gamma}_k = \Gamma_k$.

We will call the functions Γ_k *normalizing functions* for V, with respect to the given basis $\{v_k\}_k$. If we formally introduce normalized basis vectors e_k and matrix elements $\tilde{t}_{kj}$ by

$$e_k = \sqrt{\Gamma_k}\, v_k,$$

$$\Delta(e_k) = \sum_j \tilde{t}_{kj} \otimes e_j,$$

then, computing formally,

$$\tilde{t}_{kj} = \sqrt{\frac{\Gamma_k(\lambda)}{\Gamma_j(\mu)}}\, t_{kj}$$

and the unitarizability criterion can be stated as the unitarity

$$S(\tilde{t}_{kj})^* = \tilde{t}_{jk}.$$

Proposition 3.12. *When considered as elements of $\mathcal{F}_R(\mathrm{SU}(2))$, the matrix elements t_{kj}^N satisfy*

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \frac{(q^{2(k-N-\mu-1)}; q^2)_k}{(q^{-2\mu}; q^2)_k} S(t_{kj}^N)^* = \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} \frac{(q^{2(j-N-\lambda-1)}; q^2)_j}{(q^{-2\lambda}; q^2)_j} t_{jk}^N.$$

In particular, V_N is a unitarizable corepresentation of $\mathcal{F}_R(\mathrm{SU}(2))$.

To prove Proposition 3.12, we first note that

$$\sigma \circ ((* \circ S) \otimes (* \circ S)) \circ \Delta = \Delta \circ * \circ S,$$

where, as above, $\sigma(a \otimes b) = b \otimes a$. Thus, applying $\sigma \circ ((* \circ S) \otimes (* \circ S))$ to (3.10) gives

(3.15) $$\Delta(S(\gamma^{N-k}\alpha^k)^*) = \sum_{j=0}^{N} S(\gamma^{N-j}\alpha^j)^* \otimes S(t_{kj}^N)^*.$$

On the other hand, one has the identity

$$\Delta(t^N_{Nk}) = \sum_{j=0}^{N} t^N_{Nj} \otimes t^N_{jk},$$

which is a special case of (3.5). By (3.11), it may be written as

$$(3.16) \qquad \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{k(k-N)} \Delta(\alpha^k \beta^{N-k}) = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} q^{j(j-N)} \alpha^j \beta^{N-j} \otimes t^N_{jk}.$$

Comparing (3.15) and (3.16), we see that we can relate t^N_{jk} and $S(t^N_{kj})^*$ using the following lemma.

Lemma 3.13. *In the algebra* $\mathcal{F}_R(\mathrm{SU}(2))$,

$$S(\gamma^{N-k} \alpha^k)^* = C^N_k(\lambda, \mu) \alpha^k \beta^{N-k},$$

where

$$C^N_k(\lambda, \mu) = \frac{1 - q^{-2(\lambda+1)}}{1 - q^{-2(\lambda+1)+2N}} \cdot \frac{q^{k(k-N)}(q^{-2\mu}; q^2)_k}{(q^{2(k-N-\mu-1)}; q^2)_k}.$$

Proof. By definition,

$$S(\gamma)^* = F(\lambda-1)\beta, \qquad S(\alpha)^* = \frac{F(\lambda-1)}{F(\mu-1)} \alpha,$$

which gives

$$S(\gamma^{N-k}\alpha^k)^* = (F(\lambda-1)\beta)^{N-k} \left(\frac{F(\lambda-1)}{F(\mu-1)} \alpha \right)^k$$

$$= F(\lambda-1) \cdots F(\lambda-N+k) \beta^{N-k} \frac{F(\lambda-1) \cdots F(\lambda-k)}{F(\mu-1) \cdots F(\mu-k)} \alpha^k$$

$$= \frac{F(\lambda-1) \cdots F(\lambda-N)}{F(\mu-1+N-k) \cdots F(\mu-N-2k)} \beta^{N-k} \alpha^k$$

$$= \frac{F(\lambda-1) \cdots F(\lambda-N)}{F(\mu-1+N-k) \cdots F(\mu-N-2k)} \frac{q^{k(k-N)}(q^{-2\mu}; q^2)_k}{(q^{2(k-N-\mu)}; q^2)_k} \alpha^k \beta^{N-k},$$

where we used Lemma 3.1 in the last step. Inserting

$$F(\lambda-1) \cdots F(\lambda-N) = \frac{1 - q^{-2(\lambda+1)}}{1 - q^{-2(\lambda+1)+2N}}$$

and simplifying gives the desired expression. $\square$

Using this lemma we can combine (3.15) and (3.16) to the equality

$$(3.17) \qquad \begin{aligned} & \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{k(k-N)} \sum_{j=0}^{N} C^N_j(\lambda, \rho) \alpha^j \beta^{N-j} \otimes S(t^N_{kj})^* \\ & = C^N_k(\lambda, \mu) \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} q^{j(j-N)} \alpha^j \beta^{N-j} \otimes t^N_{jk}, \end{aligned}$$

with notation as in (3.9). We now observe that $C_k^N(\lambda, \mu)$ factors as a part independent of k and μ times a part independent of λ. We may therefore cancel the factors involving λ on both sides of (3.17) and move the dynamical variables to the right side of the tensor product, which gives

$$\sum_{j=0}^{N} \alpha^j \beta^{N-j} \otimes \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{k(k-N)+j(j-N)} \frac{(q^{-2\lambda}; q^2)_j}{(q^{2(j-N-\lambda-1)}; q^2)_j} S(t_{kj}^N)^*$$

$$= \sum_{j=0}^{N} \alpha^j \beta^{N-j} \otimes \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} q^{k(k-N)+j(j-N)} \frac{(q^{-2\mu}; q^2)_k}{(q^{2(k-N-\mu-1)}; q^2)_k} t_{jk}^N.$$

We want to conclude that this identity holds termwise. In view of (2.2), this follows if the family $(\alpha^j \beta^{N-j})_{j=0}^N$ is linearly independent over $\mu_r(M_{\mathfrak{h}^*})$, which is indeed true by Lemma 2.7. This completes the proof of Proposition 3.12.

Note that, in the non-dynamical case, $* \circ S$ is the (antilinear) algebra automorphism defined by $\beta \leftrightarrow \gamma$. For completeness, we also state the symmetry of the matrix elements with respect to $\alpha \leftrightarrow \delta$.

Proposition 3.14. *There is an algebra automorphism Φ of $\mathcal{F}_R(\mathrm{SL}(2))$, defined on the generators by*

$$\Phi : (\alpha, \beta, \gamma, \delta, f(\lambda), g(\mu)) \mapsto (\delta, \beta, \gamma, \alpha, f(-2-\mu), g(-2-\lambda)).$$

Moreover, $\Delta \circ \Phi = \sigma \circ (\Phi \otimes \Phi) \circ \Delta$ and

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{k(k-N)} \Phi(t_{kj}^N) = \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} q^{j(j-N)} t_{N-j,N-k}^N.$$

This can be proved similarly to Proposition 3.12, using instead of Lemma 3.13 the trivial identity $\Phi(\gamma^{N-k}\alpha^k) = \gamma^{N-k}\delta^k$. Note that since Φ interchanges the formal limits $\lambda, \mu \to \pm\infty$, it reduces to the *anti*-automorphism $\alpha \leftrightarrow \delta$ in the non-dynamical case; cf. Remark 2.10.

Remark 3.15. The symmetries Φ and $\Psi = * \circ S$ of the matrix elements permute the four parameter domains of Theorem 3.5. In fact, the symmetries can be obtained from the explicit expressions given there (note that $\Phi(\Xi) = \Psi(\Xi) = \Xi$). Conversely, after proving Theorem 3.5 in one of the four cases, we can use the symmetries to deduce the remaining three.

Combining Proposition 3.12 and (3.8) one obtains the orthogonality relations

(3.18)
$$\delta_{kl} = \sum_{j=0}^{N} (t_{jk}^N)^* \frac{\begin{bmatrix} N \\ j \end{bmatrix}_{q^2}}{\begin{bmatrix} N \\ k \end{bmatrix}_{q^2}} \frac{(q^{2(j-N-\lambda-1)}; q^2)_j (q^{-2\mu}; q^2)_k}{(q^{-2\lambda}; q^2)_j (q^{2(k-N-\mu-1)}; q^2)_k} t_{jl}^N$$

$$= \sum_{j=0}^{N} t_{kj}^N (t_{lj}^N)^* \frac{\begin{bmatrix} N \\ l \end{bmatrix}_{q^2}}{\begin{bmatrix} N \\ j \end{bmatrix}_{q^2}} \frac{(q^{2(l-N-\lambda-1)}; q^2)_l (q^{-2\mu}; q^2)_j}{(q^{-2\lambda}; q^2)_l (q^{2(j-N-\mu-1)}; q^2)_j}$$

for matrix elements. Our next goal is to find commutative versions of these identities by evaluating them in a representation of $\mathcal{F}_R(\mathrm{SU}(2))$. In fact, they will yield the

orthogonality relations for q-Racah polynomials. Thus we must first discuss dynamical representations of $\mathcal{F}_R(\mathrm{SU}(2))$.

4. Discrete orthogonality of matrix elements

4.1. **Dynamical representations of $\mathcal{F}_R(\mathrm{SL}(2))$.** We need the concept of dynamical representations from [EV2]. However, we prefer to realize these representations on vector spaces over $M_{\mathfrak{h}^*}$. Recall from §3.1 that an $\mathfrak{h}$-space V is a diagonalizable $\mathfrak{h}$-module $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, where V and V_α are vector spaces over $M_{\mathfrak{h}^*}$. Let $(D_{\mathfrak{h},V})_{\alpha\beta}$ be the space of $\mathbb{C}$-linear operators U on V such that $U(gv) = T_{-\beta}(g)U(v)$ and $U(V_\gamma) \subseteq V_{\gamma+\beta-\alpha}$ for all $g \in M_{\mathfrak{h}^*}$, $v \in V$ and $\gamma \in \mathfrak{h}^*$. Then the space $D_{\mathfrak{h},V} = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} (D_{\mathfrak{h},V})_{\alpha\beta}$ is an $\mathfrak{h}$-algebra with the moment maps $\mu_l, \mu_r \colon M_{\mathfrak{h}^*} \to (D_{\mathfrak{h},V})_{00}$ given by

$$\mu_l(f)(v) = T_{-\alpha}(f)v, \qquad \mu_r(f)(v) = fv, \quad v \in V_\alpha.$$

We define a *dynamical representation* of an $\mathfrak{h}$-algebra A on V to be an $\mathfrak{h}$-algebra homomorphism $A \to D_{\mathfrak{h},V}$. An *intertwiner* between two dynamical representations $\pi_i \colon A \to D_{\mathfrak{h},V_i}$, $i = 1, 2$, is an $\mathfrak{h}$-space morphism $\phi \colon V_1 \to V_2$ with $\phi \circ \pi_1(a) = \pi_2(a) \circ \phi$ for all $a \in A$. If V_0 is a complex subspace of V with $M_{\mathfrak{h}^*} V_0 = V$, then V_0 is a dynamical representation in the sense of [EV2], and, conversely, if V_0 is a dynamical representation in the sense of [EV2], then $M_{\mathfrak{h}^*} \otimes V_0$ is one in our sense.

Proposition 4.1. *Let, for $\mathfrak{h} = \mathbb{C}$ and $\omega \in \mathbb{C}$ arbitrary, $\mathcal{H}^\omega$ be the $\mathfrak{h}$-space with basis $\{e_k\}_{k=0}^\infty$ and the weight decomposition $\mathcal{H}^\omega = \bigoplus_{k=0}^\infty \mathcal{H}_{\omega+2k}^\omega$, $\mathcal{H}_{\omega+2k}^\omega = M_{\mathfrak{h}^*} e_k$. Then there is a dynamical representation $\pi^\omega \colon \mathcal{F}_R(\mathrm{SL}(2)) \to D_{\mathfrak{h},\mathcal{H}^\omega}$, defined on the generators by*

$$\pi^\omega(\alpha)\, ge_k = A_k(T_{-1}g)e_k, \quad \pi^\omega(\beta)\, ge_k = B_k(T_1 g)e_{k-1},$$
$$\pi^\omega(\gamma)\, ge_k = -q^{-1}(T_{-1}g)e_{k+1}, \quad \pi^\omega(\delta)\, ge_k = D_k(T_1 g)e_k,$$
$$\pi^\omega(\mu_l(f))\, ge_k = (T_{-\omega-2k}f)ge_k, \quad \pi^\omega(\mu_r(f))\, ge_k = fge_k,$$

where $g \in M_{\mathfrak{h}^*}$ is arbitrary and

$$A_k(\lambda) = q^{-k}\frac{1 - q^{2(\lambda-\omega-k+1)}}{1 - q^{2(\lambda-\omega-2k+1)}},$$

$$B_k(\lambda) = \frac{(1 - q^{2k})(1 - q^{2(\omega+k-1)})}{(1 - q^{2(\lambda+1)})(1 - q^{2(\omega+2k-\lambda-3)})}, \quad B_0(\lambda) = 0,$$

$$D_k(\lambda) = q^k\frac{1 - q^{2(\lambda+1-k)}}{1 - q^{2(\lambda+1)}}.$$

Proof. We first check that π^ω preserves the bigrading and the moment maps, and consequently the commutation relations (2.12). Then it suffices to check the first four defining relations of Definition 2.4, two relations from (2.11) and the determinant relation $c = 1$. This is straight-forward; for instance, the relation $\alpha\beta = qF(\mu-1)\beta\alpha$ is equivalent to

$$A_{k-1}(\lambda)B_k(\lambda-1) = qF(\lambda-1)B_k(\lambda)A_k(\lambda+1).$$

$\square$

The dynamical representation π^ω is irreducible, in an obvious sense, if and only if $\omega \notin \mathbb{Z}_{\leq 0}$. To see this, suppose that $U \subseteq \mathcal{H}^\omega$ is an $\mathfrak{h}$-subspace which is closed under π^ω. By definition, $U = \bigoplus_{k \in \Lambda} M_{\mathfrak{h}^*} e_k$ for some $\Lambda \subseteq \mathbb{Z}_{\geq 0}$. Since U is closed under $\pi^\omega(\gamma)$, we have $e_k \in \Lambda \Rightarrow e_{k+1} \in \Lambda$. If $\omega \notin \mathbb{Z}_{\leq 0}$, so that $B_k \neq 0$ for $k \geq 1$, we have also $e_{k+1} \in \Lambda \Rightarrow e_k \in \Lambda$ for $k \geq 0$, so we can deduce that $U = 0$ or $U = \mathcal{H}^\omega$. On the other hand, if $\omega \in \mathbb{Z}_{\leq 0}$, then the subspace $\bigoplus_{k \geq 1-\omega} M_{\mathfrak{h}^*} e_k$ is invariant.

Remark 4.2. The functions A_k, B_k, and D_k have finite limits as $(\omega - \lambda, \lambda) \to \pm\infty$ (two cases). Let us consider the case $(\omega - \lambda, \lambda) \to \infty$. On the level of the algebra, this corresponds to $\lambda \to -\infty$, $\mu \to \infty$. In view of (2.13), we let $\pi^\infty(\alpha)$, $\pi^\infty(\beta)$, $\pi^\infty(\gamma)$, and $\pi^\infty(\delta)$ be the operators on $\bigoplus_{k=0}^\infty \mathbb{C} e_k$ which are formally obtained as the limits of $\pi^\omega(\beta)$, $\pi^\omega(-q\delta)$, $\pi^\omega(\alpha)$, and $\pi^\omega(-q\gamma)$, respectively. Let us also put $f_k = (q^2; q^2)_k^{-1/2} e_k$. Then we recover the well-known $*$-representation of $\mathcal{F}_q(\mathrm{SU}(2))$ on $\ell^2(\mathbb{Z}_{\geq 0})$, cf. [VS], given by

$$\pi^\infty(\alpha)f_k = \sqrt{1 - q^{2k}} f_{k-1}, \qquad \pi^\infty(\beta)f_k = -q^{k+1} f_k,$$
$$\pi^\infty(\gamma)f_k = q^k f_k, \qquad \pi^\infty(\delta)f_k = \sqrt{1 - q^{2(k+1)}} f_{k+1}.$$

Lemma 4.3. *The element Ξ defined in* Lemma 3.3 *acts in the dynamical representation π^ω by*

$$\pi^\omega(\Xi) = (q^{\omega-1} + q^{1-\omega}) \, \mathrm{id} \, .$$

This corresponds nicely to Ξ being central. To prove the lemma, we use the first expression of (3.13). This gives

$$\pi^\omega(\Xi) \, g e_k = \big[q^{-\omega-2k+1} + q^{\omega+2k-1}$$
$$- q^{-(2\lambda-\omega-2k+2)}(1 - q^{2(\lambda-\omega-2k+2)})(1 - q^{2\lambda})(-q^{-1})B_k(\lambda - 1) \big] \, g e_k,$$

which indeed simplifies to $(q^{\omega-1} + q^{1-\omega})g e_k$.

We will now show that, in a certain sense, $(\pi^\omega)_{\omega \in \mathbb{R}}$ are unitarizable representations of $\mathcal{F}_R(\mathrm{SU}(2))$. Note that the $*$-operator on $D_{\mathfrak{h}}$, cf. §2.2, is the formal adjoint with respect to the formal inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(\lambda) \overline{g(\lambda)} \, d\lambda$ on $M_{\mathfrak{h}^*}$. Similarly, we want to find functions $\Gamma_k \in M_{\mathfrak{h}^*}$ such that

(4.1) $$\pi^\omega(x^*) = \pi^\omega(x)^*, \qquad x \in \mathcal{F}_R(\mathrm{SU}(2)),$$

where the $*$ on the right-hand side is the formal adjoint with respect to the formal pairing

$$\langle f e_k, g e_l \rangle = \delta_{kl} \int_{\mathbb{R}} f(\lambda) \overline{g(\lambda)} \, \Gamma_k(\lambda) \, d\lambda$$

on $\mathcal{H}^\omega$.

If $x \in \mathcal{F}_R(\mathrm{SU}(2))_{jk}$, so that

$$\pi^\omega(x)(g e_l) = X_l(T_{-k}g) e_{l+\frac{1}{2}(k-j)},$$
$$\pi^\omega(x^*)(g e_l) = X_l^*(T_k g) e_{l+\frac{1}{2}(j-k)}$$

for some $X_l, X_l^* \in M_{\mathfrak{h}^*}$, then

$$\langle \pi^\omega(x^*)fe_l, ge_{l+\frac{1}{2}(j-k)}\rangle = \int X_l^*(\lambda)f(\lambda+k)\overline{g(\lambda)}\,\Gamma_{l+\frac{1}{2}(j-k)}(\lambda)\,d\lambda,$$

$$\langle fe_l, \pi^\omega(x)ge_{l+\frac{1}{2}(j-k)}\rangle = \int f(\lambda)\overline{X_{l+\frac{1}{2}(j-k)}(\lambda)g(\lambda-k)}\Gamma_l(\lambda)\,d\lambda$$

$$= \int f(\lambda+k)\overline{X_{l+\frac{1}{2}(j-k)}(\lambda+k)g(\lambda)}\Gamma_l(\lambda+k)\,d\lambda,$$

so we require (recall that we write $\bar{f}(\lambda) = \overline{f(\bar\lambda)}$)

(4.2) $X_l^*(\lambda)\Gamma_{l+\frac{1}{2}(j-k)}(\lambda) = \bar{X}_{l+\frac{1}{2}(j-k)}(\lambda+k)\Gamma_l(\lambda+k).$

This gives a precise meaning to (4.1).

   Thus we must find Γ_l so that (4.2) holds for all generators. For $x = \mu_l(f)$, (4.2) holds provided that $\omega \in \mathbb{R}$. For the right moment map (4.2) is satisfied. For $x = \alpha$, $x^* = \delta$ and for $x = \beta$, $x^* = -q\gamma$, (4.2) takes the form

(4.3) $\Gamma_k(\lambda+1)A_k(\lambda+1) = \Gamma_k(\lambda)D_k(\lambda),$

(4.4) $\Gamma_{k-1}(\lambda-1)B_k(\lambda-1) = \Gamma_k(\lambda),$

where we used that $D_k = \bar{D}_k$ for $\omega \in \mathbb{R}$. Taking $k = 0$ in (4.3) shows that Γ_0 has to be 1-periodic. Iterating (4.4) then gives

$$\Gamma_k(\lambda) = \prod_{i=1}^{k} B_{k-i+1}(\lambda-i)\,\Gamma_0(\lambda-k) = \Gamma_0(\lambda)\frac{(q^2, q^{2\omega}; q^2)_k}{(q^{2(\lambda-k+1)}, q^{2(\omega-\lambda+k-1)}; q^2)_k}.$$

Choosing $\Gamma_0(\lambda) = 1$ we can immediately verify that (4.3) holds. Thus we have proved the following proposition.

Proposition 4.4. *For $\omega \in \mathbb{R}$, π^ω is, in a sense made precise above, a unitary representation of $\mathcal{F}_R(\mathrm{SU}(2))$ with respect to the formal pairing*

$$\langle fe_k, ge_l\rangle = \delta_{kl}\int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)}\,\frac{(q^2, q^{2\omega}; q^2)_k}{(q^{2(\lambda-k+1)}, q^{2(\omega-\lambda+k-1)}; q^2)_k}\,d\lambda.$$

4.2. **Discrete orthogonality relations.** We will now obtain commutative versions of the orthogonality relations (3.18), by evaluating them in a representation π^ω.

Proposition 4.5. *One has*

$$\pi^\omega(t_{kj}^N)e_m = T_{kjm}\,e_{m+j-k},$$

where $T_{kjm} = T_{kjm}^{\omega N} \in M_{\mathfrak{h}^*}$ can be expressed in terms of q-Racah polynomials (2.19) as

$$T_{kjm}^{\omega N}(\lambda) = (-1)^{j+k}q^{2k(\lambda+1)+m(N-k-j)+(k-j)(N-j+1)}\begin{bmatrix} N \\ j \end{bmatrix}_{q^2}$$

$$\times \frac{(q^{2(\lambda+1+k-j-m)}; q^2)_{N-k-j}(q^{-2(m+j)}, q^{-2(\omega-1+m+j)}; q^2)_k}{(q^{2(\lambda+1-j)}; q^2)_{N-j}(q^{2(\lambda-\omega+2+N-2j-2m)}; q^2)_k}$$

$$\times R_j(\mu(k); q^{-2(N+1)}, q^{-2(\lambda+1)}, q^{-2(m+j+1)}, q^{2(\lambda-\omega+1-m-j)}; q^2)$$

where we use the notation (2.15) *if* $N < j + k$.

Proof. We use the expressions for t_{kj}^N given in Theorem 3.5. First suppose that $k \leq j$, $k + j \leq N$, so that

$$\pi^\omega \left(t_{kj}^N \right) e_m = \pi^\omega (\gamma^{j-k} \delta^{N-k-j} P_k) e_m.$$

By Proposition 4.1 and Lemma 4.3, the element P_k acts on e_m by multiplying with an element of $M_{\mathfrak{h}^*}$ which is obtained from P_k by replacing $\mu \mapsto \lambda$, $\lambda \mapsto \lambda - \omega - 2m$, $\Xi \mapsto q^{\omega-1} + q^{1-\omega}$. Then $\pi^\omega(\gamma^{j-k} \delta^{N-k-j})$ acts by replacing $\lambda \mapsto \lambda + N - 2j$, multiplying with

$$(-q^{-1})^{j-k} \prod_{l=0}^{N-k-j-1} D_m(\lambda + l + k - j) = (-q^{-1})^{j-k} q^{m(N-k-j)} \frac{(q^{2(\lambda+1+k-j-m)}; q^2)_{N-k-j}}{(q^{2(\lambda+1+k-j)}; q^2)_{N-k-j}}$$

and shifting $e_m \mapsto e_{m+j-k}$. In conclusion, $\pi^\omega(t_{kj}^N) e_m = T \, e_{m+j-k}$, where

$$T(\lambda) = (-q^{-1})^{j-k} q^{m(N-k-j)} \frac{(q^{2(\lambda+1+k-j-m)}; q^2)_{N-k-j}}{(q^{2(\lambda+1+k-j)}; q^2)_{N-k-j}} P_k \begin{pmatrix} \lambda \mapsto \lambda-\omega+N-2j-2m \\ \mu \mapsto \lambda+N-2j \\ \Xi \mapsto q^{\omega-1}+q^{1-\omega} \end{pmatrix}$$

(4.5)
$$= (-1)^j q^{k(2\lambda-\omega+3+N+k-3j-3m)+(m-j)(N-j)-j}$$

$$\times \frac{(q^2; q^2)_{N-k}(q^{2(\lambda+1+k-j-m)}; q^2)_{N-k-j}}{(q^2; q^2)_j (q^2, q^{2(\lambda+1-j)}; q^2)_{N-j}(q^{2(\lambda-\omega+2+N-2j-2m)}; q^2)_k}$$

$$\times p_k(\tfrac{1}{2}(q^{\omega-1}+q^{1-\omega}); q^{1+\omega+2(m+j-k)}, q^{1-\omega-2m}, q^{3-\omega+2(\lambda+N-2j-m)}, q^{\omega-1-2(\lambda+k-j-m)}; q^2).$$

Writing the Askey–Wilson polynomial as a $_4\phi_3$ and inverting the order of summation, we obtain the desired expression.

In the remaining three cases of Theorem 3.5, the lemma can be proved similarly. Alternatively, one can use Proposition 3.2, which leads to a more involved computation, but allows one to treat all four cases simultaneously. $\square$

We now suppose that $\omega \in \mathbb{R}$, and write with abuse of notation

$$(4.6) \qquad \Gamma_k^N(\lambda) = \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \frac{(q^{2(k-N-\lambda-1)}; q^2)_k}{(q^{-2\lambda}; q^2)_k},$$

$$\Gamma_k^\omega(\lambda) = \frac{(q^2, q^{2\omega}; q^2)_k}{(q^{2(\lambda-k+1)}, q^{2(\omega-\lambda+k-1)}; q^2)_k}$$

for the normalizing functions of the corepresentation V_N and the representation $\mathcal{H}^\omega$; cf. Propositions 3.12 and 4.4. Since $t_{kj}^N \in \mathcal{F}_R(\mathrm{SU}(2))_{2k-N, 2j-N}$, it follows from (4.2) and Proposition 4.5 that

$$\pi^\omega((t_{kj}^N)^*) g e_m = T_{kj,m+k-j}(\lambda + 2j - N) \frac{\Gamma_m^\omega(\lambda + 2j - N)}{\Gamma_{m+k-j}^\omega(\lambda)} g(\lambda + 2j - N) e_{m+k-j}.$$

The first identity of (3.18) then gives

$$
\delta_{kl}\, e_m = \sum_{j=0}^{N} \pi^\omega \left((t_{jk}^N)^* \frac{\Gamma_j^N(\lambda)}{\Gamma_k^N(\mu)}\, t_{jl}^N \right) e_m
$$

$$
= \sum_{j=0}^{\min(N,m+l)} \pi^\omega((t_{jk}^N)^*) \frac{\Gamma_j^N(\lambda - \omega - 2m - 2l + 2j)}{\Gamma_k^N(\lambda)} T_{jlm}(\lambda) e_{m+l-j}
$$

$$
= \sum_{j=0}^{\min(N,m+l)} T_{jk,m+l-k}(\lambda + 2k - N) \frac{\Gamma_{m+l-j}^\omega(\lambda + 2k - N)}{\Gamma_{m+l-k}^\omega(\lambda)}
$$

$$
\times \frac{\Gamma_j^N(\lambda - \omega - 2m - 2l + 2j + 2k - N)}{\Gamma_k^N(\lambda + 2k - N)} T_{jlm}(\lambda + 2k - N) e_{m+l-k}.
$$

Replacing λ by $\lambda + N - 2k$ and m by $M - l$ we obtain

$$
\delta_{kl} = \sum_{j=0}^{\min(M,N)} \frac{\Gamma_{M-j}^\omega(\lambda)\Gamma_j^N(\lambda - \omega - 2M + 2j)}{\Gamma_{M-k}^\omega(\lambda + N - 2k)\Gamma_k^N(\lambda)} T_{jk,M-k}(\lambda) T_{jl,M-l}(\lambda)
$$

for $0 \le k,\, l \le \min(M,N)$. Using Proposition 4.5 we can write this explicitly as

$$
(4.7) \quad \sum_{j=0}^{\min(M,N)} \frac{1 - q^{2(\lambda-\omega+1-2M)+4j}}{1 - q^{2(\lambda-\omega+1-2M)}} \frac{(q^{2(\lambda-\omega-2M+1)}, q^{-2N}, q^{-2M}, q^{-2(\omega-1+M)}; q^2)_j}{(q^2, q^{2(\lambda-\omega+2+N-2M)}, q^{2(\lambda-\omega+2-M)}, q^{2(\lambda+1-M)}; q^2)_j}
$$

$$
\times q^{2j(\lambda+N+1)} R_k(\mu(j); q^{-2(N+1)}, q^{-2(\lambda+1)}, q^{-2(M+1)}, q^{2(\lambda-\omega+1-M)}; q^2)
$$

$$
\times R_l(\mu(j); q^{-2(N+1)}, q^{-2(\lambda+1)}, q^{-2(M+1)}, q^{2(\lambda-\omega+1-M)}; q^2)
$$

$$
= \delta_{kl} \frac{(q^{2(\omega-\lambda-1+2M-N)}, q^{-2(\lambda+N)}; q^2)_N}{(q^{2(\omega-\lambda-1+M-N)}, q^{-2(\lambda+N-M)}; q^2)_N} \frac{1 - q^{-2(\lambda+1+N)}}{1 - q^{-2(\lambda+1+N)+4k}}
$$

$$
\times \frac{(q^2, q^{-2\lambda}, q^{-2(\lambda+N-M)}, q^{2(\omega-\lambda-1+M-N)}; q^2)_k}{(q^{-2(\lambda+1+N)}, q^{-2N}, q^{-2M}, q^{-2(\omega-1+M)}; q^2)_k} q^{2k(\lambda-\omega+1-2M)},
$$

which is the orthogonality of q-Racah polynomials. In this case, $\{R_k\}_{k=0}^{\min(M,N)}$ is a complete system of orthogonal polynomials on $\{\mu(j)\}_{j=0}^{\min(M,N)}$. Similarly, the second equation in (3.18) gives the orthogonality of the dual system

$$
R_j(\mu(k); q^{-2(M+1)}, q^{2(\lambda-\omega+1-M)}, q^{-2(N+1)}, q^{-2(\lambda+1)}; q^2).
$$

In the limits $\lambda, \omega - \lambda \to \pm\infty$, cf. Remark 4.2, these relations reduce to the orthogonality of quantum q-Krawtchouk polynomials [K1].

Remark 4.6. We could have considered more general representations $\mathcal{H}^{\omega,\varepsilon}$, ω, $\varepsilon \in \mathbb{R}$, defined by the same formulas as in Proposition 4.1, but with the basis $\{e_k\}_{k\in\mathbb{Z}+\varepsilon}$. For $\varepsilon = 0$, $\mathcal{H}^\omega$ occurs as the invariant subspace spanned by $\{e_k\}_{k\in\mathbb{Z}_{\ge 0}}$. Suppose for simplicity that ε, $\omega + \varepsilon \notin \mathbb{Z}$. Then Proposition 4.4 is valid for $\mathcal{H}^{\omega,\varepsilon}$, with the convention (2.14). Working with these representations would lead to more general q-Racah polynomials, with M in (4.7) replaced by a continuous parameter.

5. Clebsch–Gordan coefficients for corepresentations

5.1. **Tensor products of $\mathfrak{h}$-coalgebroids.** Our next goal is to compute the Clebsch–Gordan coefficients of our corepresentations. We first need to discuss some additional algebraic concepts, in particular $\mathfrak{h}$-coalgebroids and their tensor products. These concepts will also play a crucial role in Appendix 1.

We define an $\mathfrak{h}$-*prealgebra* A to be a complex vector space, equipped with a bigrading $A = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} A_{\alpha\beta}$ and two left actions $\mu_l, \mu_r : M_{\mathfrak{h}^*} \to \mathrm{End}_{\mathbb{C}}(A)$ which preserve the bigrading, such that the images of μ_l and μ_r commute. We also introduce two right actions of $M_{\mathfrak{h}^*}$ on A by (2.1). A homomorphism of $\mathfrak{h}$-prealgebras is a linear map which preserves the four $M_{\mathfrak{h}^*}$-actions, or equivalently the two left actions and the bigrading.

If A and B are $\mathfrak{h}$-prealgebras we define their matrix tensor product $A\widetilde{\otimes}B$ as in §2.1. We also define another kind of tensor product $A\widehat{\otimes}B$ which is equal to the algebraic tensor product modulo the relations

(5.1) $$a\mu_l^A(f) \otimes b = a \otimes \mu_l^B(f)b, \qquad a\mu_r^A(f) \otimes b = a \otimes \mu_r^B(f)b.$$

The bigrading $A_{\alpha\beta}\widehat{\otimes}A_{\gamma\delta} \subseteq (A\widehat{\otimes}B)_{\alpha+\gamma,\,\beta+\delta}$ and the moment maps

$$\mu_l^{A\widehat{\otimes}B}(f)(a \otimes b) = \mu_l^A(f)a \otimes b,$$
$$\mu_r^{A\widehat{\otimes}B}(f)(a \otimes b) = \mu_r^A(f)a \otimes b,$$

make $A\widehat{\otimes}B$ an $\mathfrak{h}$-prealgebra. It follows from these definitions that

$$(a \otimes b)\mu_l^{A\widehat{\otimes}B}(f) = a \otimes b\mu_l^B(f),$$
$$(a \otimes b)\mu_r^{A\widehat{\otimes}B}(f) = a \otimes b\mu_r^B(f).$$

We will need the following two lemmas, the first of which is trivial.

Lemma 5.1. *If $\phi : A \to C$, $\psi : B \to D$ are homomorphisms of $\mathfrak{h}$-prealgebras, then $\phi \otimes \psi$ factors to $\mathfrak{h}$-prealgebra homomorphisms $A\widehat{\otimes}B \to C\widehat{\otimes}D$ and $A\widetilde{\otimes}B \to C\widetilde{\otimes}D$.*

Lemma 5.2. *If A, B, C, D are $\mathfrak{h}$-prealgebras, then*
$$\sigma_{23}(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$$
factors to an $\mathfrak{h}$-prealgebra homomorphism

(5.2) $$\sigma_{23} : (A\widetilde{\otimes}B)\widehat{\otimes}(C\widetilde{\otimes}D) \to (A\widehat{\otimes}C)\widetilde{\otimes}(B\widehat{\otimes}D).$$

Proof. There are two things to be checked: first, that σ_{23} maps into the subspace of $A \otimes B \otimes C \otimes D$ which maps onto $(A\widehat{\otimes}C)\widetilde{\otimes}(B\widehat{\otimes}D)$, second, that σ_{23} factors through the defining relations of $(A\widetilde{\otimes}B)\widehat{\otimes}(C\widetilde{\otimes}D)$.

For the first part, note that the left hand side of (5.2) splits into a sum of quotients of spaces of the form $A_{\alpha\beta} \otimes B_{\beta\gamma} \otimes C_{\delta\varepsilon} \otimes D_{\varepsilon\zeta}$. This component is mapped to $A_{\alpha\beta} \otimes C_{\delta\varepsilon} \otimes B_{\beta\gamma} \otimes D_{\varepsilon\zeta}$ by σ_{23}, and is then projected to $(A\widehat{\otimes}C)_{\alpha+\delta,\beta+\varepsilon} \otimes (B\widehat{\otimes}D)_{\beta+\varepsilon,\gamma+\zeta}$. Here we may indeed replace $\otimes$ by $\widetilde{\otimes}$.

For the second part, we write down the relations valid on both sides of (5.2) explicitly. Those on the left-hand side may be written as

$$\mu_r a = \mu_l b, \quad \mu_r c = \mu_l d, \quad a\mu_l = \mu_l c, \quad b\mu_r = \mu_r d,$$

where, for instance, the first identity is an abbreviation for

$$\mu_r^A(f)a \otimes b \otimes c \otimes d = a \otimes \mu_l^B(f)b \otimes c \otimes d, \quad a \in A,\, b \in B,\, c \in C,\, d \in D,\, f \in M_{\mathfrak{h}^*},$$

and those on the right-hand side as

$$\mu_r a = \mu_l b, \quad a\mu_l = \mu_l c, \quad a\mu_r = \mu_r c, \quad b\mu_l = \mu_l d, \quad b\mu_r = \mu_r d.$$

We must show that the second group of relations implies the first group. This is clear except for the relation $\mu_r c = \mu_l d$. However, by (2.3) we have also that $a\mu_r = b\mu_l$, and thus indeed $\mu_r c = a\mu_r = b\mu_l = \mu_l d$. $\qquad\square$

We define an $\mathfrak{h}$-*coalgebroid* to be an $\mathfrak{h}$-prealgebra equipped with a coproduct and a counit satisfying the same axioms as in the case of $\mathfrak{h}$-bialgebroids, except that they are required to be $\mathfrak{h}$-prealgebra homomorphisms rather than $\mathfrak{h}$-algebra homomorphisms. An $\mathfrak{h}$-coalgebroid homomorphism $\phi : A \to B$ is an $\mathfrak{h}$-prealgebra homomorphism with $(\phi \otimes \phi) \circ \Delta^A = \Delta^B \circ \phi$, $\varepsilon^B \circ \phi = \varepsilon^A$. An $\mathfrak{h}$-android is a person who studies $\mathfrak{h}$-algebroids.

Proposition 5.3. *If A and B are $\mathfrak{h}$-coalgebroids, then $\Delta^{A\widehat{\otimes}B} = \sigma_{23} \circ (\Delta^A \otimes \Delta^B)$ and $\varepsilon^{A\widehat{\otimes}B}(a \otimes b) = \varepsilon^A(a)\varepsilon^B(b)$ (a composition of difference operators) define an $\mathfrak{h}$-coalgebroid structure on $A\widehat{\otimes}B$.*

More explicitly, if $\Delta^A(a) = \sum_i a_i' \otimes a_i''$ and $\Delta^B(b) = \sum_j b_j' \otimes b_j''$, then we define

$$(5.3) \qquad \Delta^{A\widehat{\otimes}B}(a \otimes b) = \sum_{ij} a_i' \otimes b_j' \otimes a_i'' \otimes b_j''.$$

As for the proof, note that it follows from the previous two lemmas that $\Delta^{A\widehat{\otimes}B}$ is a well-defined $\mathfrak{h}$-prealgebra homomorphism. The remaining details are straightforward.

If, in particular, A is an $\mathfrak{h}$-bialgebroid, there is an $\mathfrak{h}$-algebra structure on $A\widetilde{\otimes}A$ and an $\mathfrak{h}$-coalgebroid structure on $A\widehat{\otimes}A$. One may check that the multiplication factors to an $\mathfrak{h}$-coalgebroid homomorphism $A\widehat{\otimes}A \to A$, similarly as the coproduct is an $\mathfrak{h}$-algebra homomorphism $A \to A\widetilde{\otimes}A$. In the case of bialgebras ($\mathfrak{h} = 0$) these structures combine to a bialgebra structure on $A\widetilde{\otimes}A = A\widehat{\otimes}A = A \otimes A$. By contrast, there is apparently no natural tensor product on the class of $\mathfrak{h}$-bialgebroids (that is, $\mathfrak{h}$-bialgebroids do not form a monoidal category in a natural way).

5.2. **Tensor products of corepresentations.** We will now discuss tensor products of corepresentations in general. In §3.1 we defined corepresentations of $\mathfrak{h}$-bialgebroids; this definition extends mutatis mutandis to corepresentations of $\mathfrak{h}$-coalgebroids.

When V and W are $\mathfrak{h}$-spaces we denote by $V\widehat{\otimes}W$ their tensor product over $\mathbb{C}$ modulo the relations

$$fv \otimes w = v \otimes T_\alpha fw, \qquad v \in V_\alpha.$$

The grading $V_\alpha\widehat{\otimes}W_\beta \subseteq (V\widehat{\otimes}W)_{\alpha+\beta}$ and the action of scalars $f(v \otimes w) = fv \otimes w$ make $V\widehat{\otimes}W$ into an $\mathfrak{h}$-space. Note that if we view $\mathfrak{h}$-prealgebras as $\mathfrak{h}$-spaces by "forgetting" their left (or right) moment map and grading, then the tensor product $A\widehat{\otimes}B$ of $\mathfrak{h}$-prealgebras introduced above reduces to the one defined here.

Proposition 5.4. *If V and W are corepresentation spaces of an $\mathfrak{h}$-coalgebroid A, then there is a corepresentation*

$$\pi_{V\widehat{\otimes}W} : V\widehat{\otimes}W \to A\widetilde{\otimes}(V\widehat{\otimes}W)$$

of A on on $V\widehat{\otimes}W$ defined by

(5.4) $$\pi_{V\widehat{\otimes}W} = (m \otimes \mathrm{id}) \circ \sigma_{23} \circ (\pi_V \otimes \pi_W).$$

If A is an $\mathfrak{h}$-bialgebroid, then the multiplication $m : A\widehat{\otimes}A \to A$ is an intertwiner for the regular corepresentation.

To prove the first statement, note that it follows from Lemmas 5.1 and 5.2 that $\pi_{V\widehat{\otimes}W}$ is a well-defined $\mathfrak{h}$-space morphism. The remaining details are exactly as for coalgebras. The second part of the proposition follows from the fact that Δ is an algebra homomorphism, which can be expressed as

$$\Delta \circ m = (m \otimes m) \circ \sigma_{23} \circ (\Delta \otimes \Delta).$$

This means precisely that m is an intertwiner.

If $\{v_k\}_k$, $\{w_k\}_k$ are bases (over $M_{\mathfrak{h}^*}$) of V and W, and t^V_{kj}, t^W_{kj} are matrix elements of π_V and π_W with respect to these bases, then (5.4) may be written as

$$\pi_{V\widehat{\otimes}W}(v_j \otimes w_k) = \sum_{lm} t^V_{jl}\, t^W_{km} \otimes v_l \otimes w_m,$$

or more compactly as

(5.5) $$t^{V\widehat{\otimes}W}_{jk,lm} = t^V_{jl}\, t^W_{km}.$$

Now suppose that we are given three corepresentations U, V, W and an intertwiner $C : U\widehat{\otimes}V \to W$. If we pick bases of the corepresentation spaces, the matrix elements $C_{jk,l} \in M_{\mathfrak{h}^*}$ defined by

(5.6) $$C(u_j \otimes v_k) = \sum_{l} C_{jk,l}\, w_l$$

are Clebsch–Gordan coefficients. Combining (3.7) and (5.5) gives the intertwining property in terms of these coefficients:

(5.7) $$\sum_{kl} C_{kl,p}(\mu) t^U_{mk}\, t^V_{nl} = \sum_{j} C_{mn,j}(\lambda) t^W_{jp} \quad \text{for all } m, n, p.$$

Let us now apply the coproduct Δ to (5.7). By (3.5), we obtain

$$\sum_{klxy} C_{kl,p}(\mu) t^U_{mx}\, t^V_{ny} \otimes t^U_{xk}\, t^V_{yl} = \sum_{jz} C_{mn,j}(\lambda) t^W_{jz} \otimes t^W_{zp}$$

$$= \sum_{xyz} C_{xy,z}(\rho) t^U_{mx}\, t^V_{ny} \otimes t^W_{zp},$$

where we again applied (5.7) and used the notation of (3.9). Assuming that the family $(t^U_{mx} t^V_{ny})_{xy}$ is linearly independent over $\mu_r(M_{\mathfrak{h}^*})$, it follows that

$$\sum_{kl} C_{kl,p}(\mu) t^U_{xk}\, t^V_{yl} = \sum_{z} C_{xy,z}(\lambda) t^W_{zp},$$

which is (5.7) with (m,n) replaced by (x,y). Thus, to prove that the operator C defined by (5.6) is intertwining, it suffices to check (5.7) for all p and with m and n fixed such that $(t_{mk}^U t_{nl}^V)_{kl}$ is independent over $\mu_r(M_{\mathfrak{h}^*})$.

5.3. **Clebsch–Gordan coefficients for $\mathcal{F}_R(\mathrm{SU}(2))$.** We are now ready to compute the Clebsch–Gordan coefficients of our corepresentations. The proof will be similar to the one in [KK] for the non-dynamical case.

In analogy with the classical case, we will prove that

$$\text{(5.8)} \qquad V_M \widehat{\otimes} V_N \simeq \bigoplus_{s=0}^{\min(M,N)} V_{M+N-2s}$$

(direct sums of corepresentations may be defined in a straight-forward way). Therefore we assume that $C : V_M \widehat{\otimes} V_N \to V_{M+N-2s}$ is an intertwiner, where $0 \leq s \leq \min(M,N)$. We write the Clebsch–Gordan coefficients with respect to the standard bases as

$$C(\gamma^{M-j}\alpha^j \otimes \gamma^{N-k}\alpha^k) = \sum_{l=0}^{M+N-2s} C_{jk,l}^{MN,M+N-2s}(\lambda)\gamma^{M+N-2s-l}\alpha^l.$$

Since C preserves the grading, one has $C_{jk,l}^{MN,M+N-2s} = 0$ unless $j+k = l+s$, so (5.7) may be written as

$$\text{(5.9)} \qquad \sum_{\substack{k+l=p+s \\ 0 \leq k \leq M \\ 0 \leq l \leq N}} C_{kl,p}^{MN,M+N-2s}(\mu) t_{mk}^M t_{nl}^N = C_{mn,m+n-s}^{MN,M+N-2s}(\lambda) t_{m+n-s,p}^{M+N-2s}.$$

We now observe that, by Lemma 2.7, the elements

$$t_{0k}^M t_{Nl}^N = \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} \begin{bmatrix} N \\ l \end{bmatrix}_{q^2} q^{k(k-M)+l(l-N)}\gamma^k \delta^{M-k}\alpha^l\beta^{N-l}, \quad 0 \leq k \leq M,\ 0 \leq l \leq N,$$

are linearly independent over $\mu_r(M_{\mathfrak{h}^*})$. Thus, by the observation concluding §5.2, it suffices to find C so that (5.9) holds for $m = 0$ and $n = N$, that is, so that

$$\text{(5.10)} \quad C_{0N,N-s}^{MN,M+N-2s}(\lambda) t_{N-s,p}^{M+N-2s} = \sum_{l=\max(0,\,p+s-M)}^{\min(N,\,p+s)} \begin{bmatrix} M \\ p+s-l \end{bmatrix}_{q^2} \begin{bmatrix} N \\ l \end{bmatrix}_{q^2}$$
$$\times q^{(p+s-l)(p+s-l-M)+l(l-N)} C_{p+s-l,l,p}^{MN,M+N-2s}(\mu)\gamma^{p+s-l}\delta^{M-p-s+l}\alpha^l\beta^{N-l}.$$

Note that, by Proposition 5.4, we can for $s = 0$ choose C as the multiplication $V_M \widehat{\otimes} V_N \to V_{M+N}$. One may then check that (5.10) reduces to the expression for matrix elements given in Proposition 3.2. We will in fact compute the Clebsch–Gordan coefficients by deducing an expression of the form (5.10) from Proposition 3.2. For this we need the following lemma.

Lemma 5.5. *In the algebra $\mathcal{F}_R(\mathrm{SL}(2))$,*

$$\text{(5.11)} \qquad 1 = \sum_{m=0}^{s} \begin{bmatrix} s \\ m \end{bmatrix}_{q^2} (-1)^{s-m} q^{(2m+1)(m-s)} \frac{(q^{-2\mu};q^2)_{s-m}}{(q^{-2(\mu+m+1)};q^2)_{s-m}} \gamma^{s-m}\delta^m\alpha^m\beta^{s-m}.$$

Proof. We sketch two proofs. For the first one, we make the Ansatz

$$c^s = \sum_{m=0}^{s} B_{sm}(\lambda, \mu)\gamma^{s-m}\delta^m \alpha^m \beta^{s-m},$$

where c is the dynamical determinant of Lemma 2.5. Writing

$$c^{s+1} = \left(\delta\alpha - \frac{q^{-1}}{F(\mu-1)}\gamma\beta\right)c^s = \delta c^s \alpha - \frac{q^{-1}}{F(\mu-1)}\gamma c^s \beta$$

and using the commutation relations of Lemma 3.1, one derives the recursion relations

$$B_{s+1,m}(\lambda, \mu) = q^{2(m-1-s)}\left(\frac{1-q^{-2\mu+2s-2m}}{1-q^{-2\mu-2}}\right)^2 B_{s,m-1}(\lambda+1, \mu+1)$$

$$- \frac{q^{-1}}{F(\mu-1)}B_{sm}(\lambda+1, \mu-1),$$

for $m = 0, \dots, s+1$, where $B_{s,-1} = B_{s,s+1} = 0$. This leads to an identity equivalent to (3.12).

Alternatively, one may plug the expression for $\delta^m \alpha^m$ given in Lemma 3.4 into the right-hand side of (5.11), commute γ^{s-m} across this expression (using Lemma 3.3) and then again apply Lemma 3.4 to the factor $\gamma^{s-m}\beta^{s-m}$. This results in an identity involving only commuting variables, which turns out to be the terminating $_6W_5$ summation formula [GR]:

$$1 = \frac{(q^{1+\lambda-\mu}\xi, q^{1+\lambda-\mu}\xi^{-1}; q^2)_s}{(q^{-2(\mu+1)}, q^{2(\lambda+2)}; q^2)_s} {}_6W_5(q^{2(1+\mu-s)}; q^{\lambda+\mu+3}\xi, q^{\lambda+\mu+3}\xi^{-1}, q^{-2s}; q^2, q^{-2(\lambda+1)}),$$

with ξ a formal quantity satisfying $\xi + \xi^{-1} = \Xi$. $\qquad\square$

To compute the Clebsch–Gordan coefficients we insert the right-hand side of (5.11) into the expressions for matrix elements given in Proposition 3.2. Using Lemma 3.1 twice and pulling the dynamical variables to the left gives

$$t_{N-s,p}^{M+N-2s} = \sum_{l=\max(0,\,p+s-M)}^{\min(N-s,\,p)} \begin{bmatrix} M-s \\ p-l \end{bmatrix}_{q^2} \begin{bmatrix} N-s \\ l \end{bmatrix}_{q^2} q^{p(p+N-M)+l(3l+M+s-2N-3p)}$$

$$\times \frac{(q^{2(p+2s-M-N-\mu-1)}; q^2)_{p-l}}{(q^{2(p+s-M-l-\mu-1)}; q^2)_{p-l}}\gamma^{p-l}\delta^{M-p-s+l}\left(\sum_{m=0}^{s}\begin{bmatrix} s \\ m \end{bmatrix}_{q^2}(-1)^{s-m}q^{(2m+1)(m-s)}\right.$$

$$\times \left. \frac{(q^{-2\mu}; q^2)_{s-m}}{(q^{-2(\mu+m+1)}; q^2)_{s-m}}\gamma^{s-m}\delta^m \alpha^m \beta^{s-m}\right)\alpha^l \beta^{N-s-l}$$

$$= \sum_{l=\max(0,\,p+s-M)}^{\min(N-s,p)} \sum_{m=0}^{s} \begin{bmatrix} M-s \\ p-l \end{bmatrix}_{q^2} \begin{bmatrix} N-s \\ l \end{bmatrix}_{q^2} \begin{bmatrix} s \\ m \end{bmatrix}_{q^2} (-1)^{s-m}$$

$$\times q^{p(p+N-M)+l(3l+M+s-2N-3p)+(M-p-s+2l+2m+1)(m-s)}$$

$$\times \frac{(q^{2(p+2s-M-N-\mu-1)};q^2)_{p-l}\big(q^{2(2p+s-M-l-\mu)},q^{2(p-l-\mu)};q^2\big)_{s-m}}{(q^{2(p+s-M-l-\mu-1)};q^2)_{p-l}\big(q^{2(2p+s-M-2l-m-\mu-1)},q^{2(2p+s-M-2l-\mu)};q^2\big)_{s-m}}$$

$$\times \gamma^{p+s-l-m}\delta^{M-p-s+l+m}\alpha^{l+m}\beta^{N-l-m}.$$

Replacing l by $l - m$, we see that (5.10) holds with

$$(5.12) \quad C_{jk,j+k-s}^{MN,M+N-2s}(\lambda) = \sum_{m=\max(0,s-j,k+s-N)}^{\min(s,k,M-j)} \frac{\begin{bmatrix} M-s \\ j+m-s \end{bmatrix}_{q^2} \begin{bmatrix} N-s \\ k-m \end{bmatrix}_{q^2} \begin{bmatrix} s \\ m \end{bmatrix}_{q^2}}{\begin{bmatrix} M \\ j \end{bmatrix}_{q^2} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2}} (-1)^{s+m}$$

$$\times q^{(N-k-s)(j-s)-s+m(1+2j+2N+3m-2k-4s)}$$

$$\times \frac{(q^{2(j+k+s-M-N-\lambda-1)};q^2)_{j+m-s}\big(q^{2(2j+k+m-s-M-\lambda)},q^{2(j+m-s-\lambda)};q^2\big)_{s-m}}{(q^{2(j+m-M-\lambda-1)};q^2)_{j+m-s}\big(q^{2(2j+2m-s-M-\lambda)},q^{2(2j+m-s-M-\lambda-1)};q^2\big)_{s-m}}$$

(in particular, the Clebsch–Gordan coefficient on the left-hand side of (5.10) reduces to 1). For all values of the parameters, this is an $_8W_7$-sum; for instance, when $s \le \min(j, N-k)$ we get

$$C_{jk,j+k-s}^{MN,M+N-2s}(\lambda) = (-1)^s q^{(j-s)(N-k-s)-s} \frac{(q^2;q^2)_{M-s}(q^2;q^2)_{N-s}(q^2;q^2)_j(q^2;q^2)_{N-k}}{(q^2;q^2)_M(q^2;q^2)_N(q^2;q^2)_{j-s}(q^2;q^2)_{N-k-s}}$$

$$\times \frac{(q^{2(j+k+s-M-N-\lambda-1)};q^2)_{j-s}\big(q^{2(2j+k-s-M-\lambda)},q^{2(j-s-\lambda)};q^2\big)_s}{(q^{2(j-M-\lambda-1)};q^2)_j(q^{2(2j-s-M-\lambda)};q^2)_s} \,_8W_7\big(q^{2(2j-s-M-\lambda-1)};$$

$$q^{-2s},q^{2(j-M-\lambda-1)},q^{-2k},q^{2(j-M)},q^{2(2j+k-M-N-\lambda-1)};q^2,q^{2(2+M+N-s)}\big).$$

Using (2.17) we can write this as

$$(5.13) \quad\begin{aligned} C_{jk,j+k-s}^{MN,M+N-2s}(\lambda) &= q^{(j-s)(N-k)+sj} \frac{(q^{-2(j+k)};q^2)_s(q^{2(j+k-M-N-\lambda-1)};q^2)_j}{(q^{-2N};q^2)_s(q^{2(j-M-\lambda-1)};q^2)_j} \\ &\quad \times \,_4\phi_3\begin{bmatrix} q^{-2s},q^{2(s-M-N-1)},q^{-2j},q^{2(j-M-\lambda-1)} \\ q^{-2M},q^{-2(j+k)},q^{2(j+k-M-N-\lambda-1)} \end{bmatrix};q^2,q^2\end{bmatrix}. \end{aligned}$$

It is easy to verify that (5.13) holds also without the assumption $s \le \min(j, N-k)$.

For each $s \le \min(M, N)$, we have constructed an intertwiner $C_s : V_M \widehat{\otimes} V_N \to V_{M+N-2s}$, which is clearly surjective. Since both sides of (5.8) have the same dimension, $\oplus_s C_s$ is a bijective intertwiner from the left-hand to the right-hand side. Thus we have proved the following theorem.

Theorem 5.6. *The decomposition* (5.8) *holds as an equivalence of corepresentations. The equation*

$$(5.14) \qquad C_s(\gamma^{M-j}\alpha^j \otimes \gamma^{N-k}\alpha^k) = C_{jk,j+k-s}^{MN,M+N-2s}(\lambda)\gamma^{M+N-s-j-k}\alpha^{j+k-s},$$

where the coefficients are given in (5.13), *defines an intertwiner* $C_s : V_M \widehat{\otimes} V_N \to V_{M+N-2s}$. *In particular, the Clebsch–Gordan formula* (5.9) *holds.*

5.4. **Orthogonality of the Clebsch–Gordan coefficients.** We will now discuss the orthogonality of Clebsch–Gordan coefficients, which is a consequence of Schur's Lemma (Corollary 3.9) and the unitarizability of the corepresentations (Proposition 3.12). We will see that it yields orthogonality relations for q-Racah polynomials. We start with two general facts about unitarizable corepresentations.

Lemma 5.7. *Let V_1 and V_2 be two unitarizable corepresentation spaces of a $*$-$\mathfrak{h}$-Hopf algebroid A, and let Γ_k^i, $i = 1, 2$, be normalizing functions with respect to some bases $\{v_k^i\}_k$ of V_i.*

If $\phi : V_1 \to V_2$ is an intertwiner, and $\phi_{kj} \in M_{\mathfrak{h}^*}$ the matrix elements of ϕ with respect to the bases $\{v_k^i\}_k$, as in (3.6), then $\phi_{kj}^* = (\Gamma_j^1/\Gamma_k^2)\bar{\phi}_{jk}$ defines an intertwiner $\phi^* : V_2 \to V_1$.

Moreover, the tensor product corepresentation $V_1 \widehat{\otimes} V_2$ is unitarizable, with

$$\Gamma_{jk}(\lambda) = \Gamma_j^1(\lambda)\Gamma_k^2(\lambda - \omega(j))$$

a normalizing function with respect to the basis $\{v_j^1 \otimes v_k^2\}_{jk}$, where $\omega(j) \in \mathfrak{h}^*$ is defined by $v_j^1 \in (V_1)_{\omega(j)}$.

Proof. For the first statement, we apply $* \circ S$ to (3.7). Using the unitarizability we obtain

$$\sum_j \bar{\phi}_{jl}(\lambda)\frac{\Gamma_j^1(\lambda)}{\Gamma_k^1(\mu)} t_{jk}^1 = \sum_j \bar{\phi}_{kj}(\mu)\frac{\Gamma_l^2(\lambda)}{\Gamma_j^2(\mu)} t_{lj}^2 \quad \text{for all } k, l,$$

which means precisely that ϕ^* is intertwining.

For the second statement, we apply $* \circ S$ to (5.5). This gives indeed

$$S(t_{jk,lm}^{V_1 \widehat{\otimes} V_2})^* = \frac{\Gamma_l^1(\lambda)}{\Gamma_j^1(\mu)} t_{lj}^1 \frac{\Gamma_m^2(\lambda)}{\Gamma_k^2(\mu)} t_{mk}^2 = \frac{\Gamma_l^1(\lambda)\Gamma_m^2(\lambda - \omega(l))}{\Gamma_j^1(\mu)\Gamma_k^2(\mu - \omega(j))} t_{lm,jk}^{V_1 \widehat{\otimes} V_2},$$

where we used that $t_{lj}^1 \in A_{\omega(l),\omega(j)}$. $\qquad\square$

In view of the second part of Lemma 5.7, we may apply the first part to the intertwiner $C_s : V_M \widehat{\otimes} V_N \to V_{M+N-2s}$ defined in (5.14). This yields an intertwiner $C_s^* : V_{M+N-2s} \to V_M \widehat{\otimes} V_N$, defined by

$$C_s^*(\gamma^{M+N-2s-l}\alpha^l) = \sum_{\substack{j+k=l+s \\ 0 \le j \le M \\ 0 \le k \le N}} \frac{\Gamma_j^M T_{M-2j}\Gamma_k^N}{\Gamma_l^{M+N-2s}} C_{jk,j+k-s}^{MN,M+N-2s}\gamma^{M-j}\alpha^j \otimes \gamma^{N-k}\alpha^k,$$

where the Clebsch–Gordan coefficients are given by (5.13) and the normalizing functions by (4.6).

Let us now consider the map $C_s C_t^* : V_{M+N-2t} \to V_{M+N-2s}$, where $0 \le s, t \le \min(M, N)$. By Corollary 3.9,

$$(5.15) \qquad\qquad\qquad C_s C_t^* = \delta_{st} Z_s \, \mathrm{id}$$

for some $Z_s \in \mathbb{C}$. Applying this identity to $\gamma^{M+N-t-L}\alpha^{L-t}$ gives the orthogonality relations for Clebsch–Gordan coefficients:

$$(5.16) \qquad \delta_{st} Z_s = \sum_{\substack{j+k=L \\ 0 \le j \le M \\ 0 \le k \le N}} \frac{\Gamma_j^M(\lambda)\Gamma_k^N(\lambda+M-2j)}{\Gamma_{L-t}^{M+N-2t}(\lambda)} C_{jk,L-s}^{MN,M+N-2s}(\lambda) C_{jk,L-t}^{MN,M+N-2t}(\lambda),$$

where $0 \le s, t \le \min(L, M, N, M+N-L)$.

To compute Z_s, we specialize (5.16) to the case $L = s = t$. Then the Clebsch–Gordan coefficients on the right-hand side simplify, since the sum in (5.12) reduces to the term with $m = k$, giving

$$(5.17) \quad C_{jk,0}^{MN,M+N-2s}(\lambda) = (-1)^j q^{k(N-s)-j} \frac{(q^2;q^2)_s (q^2;q^2)_{M-j}(q^2;q^2)_{N-k}(q^{-2\lambda};q^2)_j}{(q^2;q^2)_M (q^2;q^2)_N (q^{2(j-M-\lambda-1)};q^2)_j}$$

for $j + k = s$. Plugging in this expression and simplifying, the right-hand side of (5.16) reduces to a $_6W_5$ sum:

$$Z_s = (-1)^s q^{-s(s+1)} \frac{(q^2, q^{2(s-M-N-\lambda-1)};q^2)_s}{(q^{-2N}, q^{-2(\lambda+M)};q^2)_s}$$
$$\times\ _6W_5\big(q^{-2(\lambda+1+M)}; q^{-2s}, q^{2(1+N-s)}, q^{-2\lambda}; q^2, q^{2(2s-M-N-1)}\big)$$
$$(5.18) \qquad = (-1)^s q^{-s(s+1)} \frac{(q^2, q^{2(s-M-N-1)};q^2)_s}{(q^{-2M}, q^{-2N};q^2)_s},$$

where the second step is obtained either by applying [GR, (II.21)] or by observing that, since we know a priori that Z_s is independent of λ, we can put $\lambda = 0$, so that the $_6W_5$ reduces to 1.

Using (5.13) one may check that (5.16) is the orthogonality of the q-Racah polynomials

$$(5.19) \qquad R_s(\mu(j); q^{-2(M+1)}, q^{-2(N+1)}, q^{-2(L+1)}, q^{2(L-M-\lambda-1)}; q^2),$$

cf. (2.19). For this special case, $\{R_s\}_{s=0}^{\min(L,M,N,M+N-L)}$ is, for generic λ, a complete system of polynomials orthogonal on $\{\mu(j)\}_{j=\max(0,L-N)}^{\min(L,M)}$.

To obtain the dual orthogonality relations, we observe that

$$(5.20) \qquad \mathrm{id}\big|_{V_M \widehat{\otimes} V_N} = \sum_{s=0}^{\min(M,N)} \frac{1}{Z_s} C_s^* C_s.$$

In fact, it follows from (5.15) that the restriction of both sides to the image of C_t^* are equal for $0 \le t \le \min(M, N)$. By Theorem 5.6, these images span $V_m \widehat{\otimes} V_N$. Applying (5.20) to a tensor product $\gamma^{M-j}\alpha^j \otimes \gamma^{N-L+j}\alpha^{L-j}$ gives

$$(5.21) \qquad \delta_{jk} = \sum_{s=0}^{\min(L,M,N,M+N-L)} \frac{1}{Z_s} \frac{\Gamma_j^M(\lambda)\Gamma_{L-j}^N(\lambda+M-2j)}{\Gamma_{L-s}^{M+N-2s}(\lambda)}$$
$$\times\ C_{j,L-j,L-s}^{MN,M+N-2s}(\lambda) C_{k,L-k,L-s}^{MN,M+N-2s}(\lambda),$$

for $\max(0, L - n) \leq j, k \leq \min(L, M)$. This is the orthogonality of the system

$$R_j(\mu(s); q^{-2(L+1)}, q^{2(L-M-\lambda-1)}, q^{-2(M+1)}, q^{-2(N+1)}; q^2)$$

dual to (5.19). In the non-dynamical limit, (5.16) and (5.21) are orthogonality relations for q-Hahn and dual q-Hahn polynomials, respectively; cf. [KK].

Another consequence of (5.20) is the Clebsch–Gordan formula dual to (5.9).

Proposition 5.8. *The identity*

$$
(5.22) \qquad t_{kx}^M \, t_{ly}^N = \sum_s \frac{1}{Z_s} \frac{\Gamma_x^M(\mu)\Gamma_y^N(\mu + M - 2x)}{\Gamma_{x+y-s}^{M+N-2s}(\mu)}
$$
$$
\times C_{kl,k+l-s}^{MN,M+N-2s}(\lambda) \, C_{xy,x+y-s}^{MN,M+N-2s}(\mu) \, t_{k+l-s,x+y-s}^{M+N-2s},
$$

holds, where $0 \leq k, x \leq M$, $0 \leq l, y \leq N$, *the sum runs over s with*

$$0 \leq s \leq \min(M, N, k + l, M + N - k - l, x + y, M + N - x - y),$$

and Z_s is given by (5.18).

This is proved by applying (5.20) to $\pi^{V_M \widehat{\otimes} V_N}(\gamma^{M-k}\alpha^k \otimes \gamma^{N-l}\alpha^l)$, using the intertwining property of C_s, and identifying the coefficient of $\gamma^{M-x}\alpha^x \otimes \gamma^{N-y}\alpha^y$. If we apply the counit ε to (5.22), we recover (5.21).

One can obtain commutative versions of the Clebsch–Gordan formulas (5.9) and (5.22) by evaluating them in a dynamical representation. Namely, applying the representation π^ω of Proposition 4.1 to (5.9) and acting with both sides on $e_j \in \mathcal{H}^\omega$ gives the identity

$$
(5.23) \qquad \sum_{k+l=p+s} C_{kl,p}^{MN,M+N-2s}(\lambda) \, T_{mk,j+l-n}^{\omega M}(\lambda) \, T_{nlj}^{\omega N}(\lambda + M - 2k)
$$
$$
= C_{mn,m+n-s}^{MN,M+N-2s}(\lambda - \omega - 2(j + p + s - m - n)) \, T_{m+n-s,pj}^{\omega,M+N-2s}(\lambda),
$$

where $T_{kjm}^{\omega N}$ is the function from Proposition 4.5, while (5.22) similarly gives

$$
(5.24) \quad T_{kx,j+y-l}^{\omega M}(\lambda) \, T_{lyj}^{\omega N}(\lambda + M - 2x) = \sum_s \frac{1}{Z_s} \frac{\Gamma_x^M(\lambda)\Gamma_y^N(\lambda + M - 2x)}{\Gamma_{x+y-s}^{M+N-2s}(\lambda)}
$$
$$
\times C_{kl,k+l-s}^{MN,M+N-2s}(\lambda - \omega - 2(j + x + y - k - l)) \, C_{xy,x+y-s}^{MN,M+N-2s}(\lambda) \, T_{k+l-s,x+y-s,j}^{\omega,M+N-2s}(\lambda).
$$

Using Proposition 4.5 and (5.13) we may express these identities in terms of q-Racah polynomials, as indicated in (4.7) and (5.19). It turns out that in both cases we obtain instances of the Biedenharn–Elliott identity, which will be discussed in more detail in §6.2. In the non-dynamical case, (5.23) and (5.24) are essentially different identities, cf. [KV1, KV2] for related results.

6. Clebsch–Gordan coefficients for representations

6.1. **Tensor products of dynamical representations.** In this section we will obtain the Clebsch–Gordan decomposition of the dynamical representations introduced in §4.1. Since our definition of dynamical representations differs slightly from the

one in [EV2], we must accordingly modify the definition of tensor product representations.

When V and W are $\mathfrak{h}$-spaces, we denote by $V\bar{\otimes}W$ their tensor product over $\mathbb{C}$ modulo the relations

$$fv \otimes w = v \otimes T_{-\beta}fw, \quad w \in W_\beta.$$

The grading $V_\alpha\bar{\otimes}W_\beta \subseteq (V\bar{\otimes}W)_{\alpha+\beta}$ and the action of scalars $f(v \otimes w) = v \otimes fw$ make $V\bar{\otimes}W$ into an $\mathfrak{h}$-space. This is closely related to the tensor product $V\widehat{\otimes}W$ introduced in §5.2; in fact, the flip map $v\otimes w \mapsto w\otimes v$ defines an $\mathfrak{h}$-space isomorphism $V\bar{\otimes}W \mapsto W\widehat{\otimes}V$.

Let $\pi_V : A \to D_{\mathfrak{h},V}$ and $\pi_W : A \to D_{\mathfrak{h},W}$ be two dynamical representations of an $\mathfrak{h}$-algebra A. One may check that the identity operator factors to an $\mathfrak{h}$-algebra homomorphism $\Theta : D_{\mathfrak{h},V}\widetilde{\otimes}D_{\mathfrak{h},W} \to D_{\mathfrak{h},V\bar{\otimes}W}$. Then

$$\pi_{V\bar{\otimes}W} = \Theta \circ (\pi_V \otimes \pi_W) \circ \Delta$$

defines a dynamical representation of A on $V\bar{\otimes}W$.

We will obtain the decomposition

$$(6.1) \qquad \mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2} \simeq \bigoplus_{s=0}^{\infty} \mathcal{H}^{\omega_1+\omega_2+2s}$$

(direct sums of dynamical representations may be defined in a straight-forward way). In view of Lemma 4.3, we can achieve this by diagonalizing the action of Ξ in the tensor product representation $\pi = \pi_{\mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2}}$. Note the resemblance of (6.1) to the case of highest weight (discrete series) representations of $\mathfrak{su}(1,1)$. In this analogy, Ξ corresponds to the Casimir operator and ω_i to the highest weights.

From (3.13) we see that we first have to consider

$$\Delta(\gamma\beta) = \gamma\alpha \otimes \alpha\beta + \gamma\beta \otimes \alpha\delta + \delta\alpha \otimes \gamma\beta + \delta\beta \otimes \gamma\delta,$$

where we used Definition 2.4. Thus, $\pi(\gamma\beta)(e_{k_1} \otimes e_{k_2})$ can be written as a sum of four terms, the first of which is

$$\begin{aligned}
\pi^{\omega_1}(\gamma\alpha)e_{k_1} \otimes \pi^{\omega_2}(\alpha\beta)e_{k_2} &= -q^{-1}(T_{-1}A_{k_1}^{\omega_1})e_{k_1+1} \otimes A_{k_2-1}^{\omega_2}(T_{-1}B_{k_2}^{\omega_2})e_{k_2-1} \\
&= -q^{-1}(T_{-(\omega_2+2(k_2-1))-1}A_{k_1}^{\omega_1})A_{k_2-1}^{\omega_2}(T_{-1}B_{k_2}^{\omega_2})(e_{k_1+1} \otimes e_{k_2-1}),
\end{aligned}$$

where we have written ω_i as superscripts to the functions A_k and B_k from Proposition 4.1 to indicate their dependence on ω. Computing the other three terms similarly gives

$$(6.2) \qquad \begin{aligned}
\pi(\gamma\beta)\,e_{k_1} \otimes e_{k_2} &= a_{k_1,k_2}(e_{k_1+1} \otimes e_{k_2-1}) + b_{k_1,k_2}(e_{k_1} \otimes e_{k_2}) \\
&\quad + c_{k_1,k_2}(e_{k_1-1} \otimes e_{k_2+1}),
\end{aligned}$$

with

$$-qa_{k_1,k_2}(\lambda) = B_{k_2}^{\omega_2}(\lambda-1)A_{k_2-1}^{\omega_2}(\lambda)A_{k_1}^{\omega_1}(\lambda-\omega_2-2k_2+1),$$
$$-qb_{k_1,k_2}(\lambda) = D_{k_2}^{\omega_2}(\lambda-1)A_{k_2}^{\omega_2}(\lambda)B_{k_1}^{\omega_1}(\lambda-\omega_2-2k_2-1)$$
$$\qquad + B_{k_2}^{\omega_2}(\lambda-1)A_{k_1}^{\omega_1}(\lambda-\omega_2-2k_2+1)D_{k_1}^{\omega_1}(\lambda-\omega_2-2k_2),$$
$$-qc_{k_1,k_2}(\lambda) = D_{k_2}^{\omega_2}(\lambda-1)B_{k_1}^{\omega_1}(\lambda-\omega_2-2k_2-1)D_{k_1-1}^{\omega_1}(\lambda-\omega_2-2k_2-2).$$

From (6.2) or from $\gamma\beta \in \mathcal{F}_R(\mathrm{SL}(2))_{00}$ it follows that $\pi(\gamma\beta)$ preserves the weight spaces $(\mathcal{H}^{\omega_1}\bar\otimes\mathcal{H}^{\omega_2})_{\omega_1+\omega_2+2p} = \bigoplus_{k_1+k_2=p}\mathcal{H}_{\omega_1+2k_1}^{\omega_1}\bar\otimes\mathcal{H}_{\omega_1+2k_2}^{\omega_2}$, so that in order to diagonalize $\pi(\gamma\beta)$ it suffices to diagonalize $\pi(\gamma\beta)$ in every weight space. Fix $p \in \mathbb{Z}_{\geq 0}$; then it follows from (6.2) that $\sum_{k=0}^{p} v_k(e_{p-k}\otimes e_k)$, $v_k \in M_{\mathfrak{h}^*}$, is an eigenvector of $\pi(\gamma\beta)$ in the weight space $(\mathcal{H}^{\omega_1}\bar\otimes\mathcal{H}^{\omega_2})_{\omega_1+\omega_2+2p}$ with eigenvalue x if and only if the v_k's satisfy

(6.3) $$x\,v_k = a_{p-k-1,k+1}v_{k+1} + b_{p-k,k}v_k + c_{p-k+1,k-1}v_{k-1},$$

where $v_{-1} = v_{p+1} = 0$.

The three-term recurrence relation (6.3) can be solved in terms of q-Racah polynomials. We recall that the polynomials $R_n(\mu(x)) = R_n(\mu(x); a,b,c,d;q)$, cf. (2.19), satisfy the recurrence

(6.4) $$\big(\mu(x) - \mu(0)\big)R_n(\mu(x)) = A_n\big(R_{n+1}(\mu(x)) - R_n(\mu(x))\big)$$
$$\qquad\qquad + C_n\big(R_{n-1}(\mu(x)) - R_n(\mu(x))\big),$$

where

$$A_n = \frac{(1-abq^{n+1})(1-aq^{n+1})(1-bdq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$
$$C_n = \frac{q(1-q^n)(1-bq^n)(c-abq^n)(d-aq^n)}{(1-abq^{2n})(1-abq^{2n+1})},$$

which holds for $n, x \in \{0, 1, \dots, N\}$ if aq, bdq or cq equals q^{-N}, $N \in \mathbb{Z}_{\geq 0}$.

Upon replacing in (6.3)

$$v_k = (-1)^k q^{2k(\omega_1+\omega_2)+k(3p-1)}\frac{(q^{-2\lambda}, q^{-2(\omega_1+p-1)}, q^{-2p}; q^2)_k}{(q^2, q^{2\omega_2}, q^{2(\omega_1+\omega_2+p-\lambda-1)}; q^2)_k}\, R_k,$$

we find after a straightforward calculation that the resulting three-term recurrence relation can be written as

(6.5) $$q^{1-2(2p+\omega_1+\omega_2-1)}(1-q^{2\lambda})(1-q^{2(2p+\omega_1+\omega_2-\lambda-2)})\,xR_k$$
$$= a_k(R_{k+1} - R_k) + c_k(R_{k-1} - R_k),$$

where

$$a_k = \frac{(1-q^{2(k+\omega_2-\lambda-1)})(1-q^{2(k-\lambda)})(1-q^{2(k-\omega_1-p+1)})(1-q^{2(k-p)})}{(1-q^{2(2k+\omega_2-\lambda-1)})(1-q^{2(2k+\omega_2-\lambda)})},$$
$$c_k = \frac{q^2(1-q^{2k})(1-q^{2(k+\omega_2-1)})(q^{-2(p+1)}-q^{2(k+\omega_2-\lambda-2)})(q^{-2(\omega_1+\omega_2+p-1)}-q^{2(k-\lambda-1)})}{(1-q^{2(2k+\omega_2-\lambda-2)})(1-q^{2(2k+\omega_2-\lambda-1)})}.$$

The right-hand side of (6.5) is the right-hand side of (6.4) in base q^2 with $a \mapsto q^{-2(\lambda+1)}$, $b \mapsto q^{2(\omega_2-1)}$, $c \mapsto q^{-2(p+1)}$ and $d \mapsto q^{-2(\omega_1+\omega_2+p-1)}$, so that $N = p$. Comparing the left-hand sides, one finds that the eigenvalue x is of the form

$$x = -q^{2(\omega_1+\omega_2+2p)-3} \frac{(1-q^{-2z})(1-q^{2(z+1-\omega_1-\omega_2-2p)})}{(1-q^{2\lambda})(1-q^{2(2p+\omega_1+\omega_2-\lambda-2)})}$$

for $z \in \{0, 1, \ldots, p\}$. The corresponding eigenvalue of $\pi(\Xi)$ can then be computed from (3.13) using the fact that we restrict to the weight space $(\mathcal{H}^{\omega_1} \bar{\otimes} \mathcal{H}^{\omega_2})_{\omega_1+\omega_2+2p}$:

$$q^{1-\omega_1-\omega_2-2p} + q^{\omega_1+\omega_2+2p-1} - q^{-(2\lambda-\omega_1-\omega_2-2p+2)}(1-q^{2(\lambda-\omega_1-\omega_2-2p+2)})(1-q^{2\lambda})x$$
$$= q^{\omega_1+\omega_2+2(p-z)-1} + q^{1-\omega_1-\omega_2-2(p-z)}$$

for $z \in \{0, 1, \ldots, p\}$. Thus we have proved the following proposition.

Proposition 6.1. *In the tensor product representation $\pi = \pi_{\mathcal{H}^{\omega_1} \bar{\otimes} \mathcal{H}^{\omega_2}}$ the element Ξ has eigenvectors $v(y; p) \in (\mathcal{H}^{\omega_1} \bar{\otimes} \mathcal{H}^{\omega_2})_{\omega_1+\omega_2+2p}$, $y \in \{0, 1, \ldots, p\}$, with the eigenvalue $q^{\omega_1+\omega_2+2y-1} + q^{1-\omega_1-\omega_2-2y}$. The eigenvector $v(y; p)$ is given in terms of q-Racah polynomials (2.19) by*

$$v(y; p) = \sum_{k=0}^{p} v_k(e_{p-k} \otimes e_k),$$

$$v_k(\lambda) = (-1)^k q^{2k(\omega_1+\omega_2)+k(3p-1)} \frac{(q^{-2\lambda}, q^{-2(\omega_1+p-1)}, q^{-2p}; q^2)_k}{(q^2, q^{2\omega_2}, q^{2(\omega_1+\omega_2+p-\lambda-1)}; q^2)_k}$$
$$\times R_k(\mu(p-y); q^{-2(\lambda+1)}, q^{2(\omega_2-1)}, q^{-2(p+1)}, q^{-2(\omega_1+\omega_2+p-1)}; q^2).$$

Note that for $\omega_1 + \omega_2 \notin \mathbb{Z}_{\leq 0}$ the eigenvalues per weight space are different and independent of λ, so that the eigenvectors are linearly independent over $M_{\mathfrak{h}^*}$ and form a basis for the tensor product representation space.

From now on we assume the genericity condition $\omega_1 + \omega_2 \notin \mathbb{Z}_{\leq 0}$. We need to calculate $\pi(\gamma)v(y; p)$. Since $\pi(\gamma)$ commutes with $\pi(\Xi)$ and raises the degree by 2, we have $\pi(\gamma)v(y; p) = Cv(y; p+1)$ for some $C \in M_{\mathfrak{h}^*}$. On the other hand, using $\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma$ we find that $\pi(\gamma)v(y; p)$ equals

$$-q^{-1} \sum_{k=0}^{p} v_k(\lambda-1) \left(A_k^{\omega_2}(\lambda)(e_{p-k+1} \otimes e_k) + D_{p-k}^{\omega_1}(\lambda - \omega_2 - 2k - 2)(e_{p-k} \otimes e_{k+1}) \right).$$

Comparing the coefficient of $e_{p+1} \otimes e_0$ yields $C(\lambda) = -q^{-1}$ or

(6.6) $$\pi(\gamma)v(y; p) = -q^{-1}v(y; p+1).$$

A similar computation gives

(6.7) $$\pi(\alpha)v(y; p) = q^{-p} \frac{1 - q^{2(\lambda-\omega_1-\omega_2-p+1)}}{1 - q^{2(\lambda-\omega_1-\omega_2-2p+1)}} v(y; p).$$

We can now construct an intertwiner $C: \mathcal{H}^{\omega_1+\omega_2+2s} \to \mathcal{H}^{\omega_1} \bar{\otimes} \mathcal{H}^{\omega_2}$. Since C must preserve the grading and the eigenspaces of Ξ, one has

$$Ce_k = \phi_k\, v(s; s+k)$$

for some $\phi_k \in M_{\mathfrak{h}^*}$. Then the intertwining property

(6.8) $$C \circ \pi_{\mathcal{H}^{\omega_1+\omega_2+2s}}(x) = \pi_{\mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2}}(x) \circ C,$$

is automatically satisfied for $x = \mu_l(f)$, $x = \mu_r(f)$ and $x = \Xi$, and thus for $x \in \mathcal{F}_R(\mathrm{SL}(2))_{00}$. Using (6.7) and (6.6) we write out (6.8) explicitly for $x = \alpha$ and $x = \gamma$. This leads to the equations

$$\phi_k(\lambda) = q^{-s}\frac{1 - q^{2(\lambda-\omega_1-\omega_2-s-k+1)}}{1 - q^{2(\lambda-\omega_1-\omega_2-2s-k+1)}}\,\phi_k(\lambda - 1)$$
$$\phi_{k-1}(\lambda) = \phi_k(\lambda + 1),$$

which are solved by $\phi_k(\lambda) = q^{s(k-\lambda)}(q^{2(\lambda+2-\omega_1-\omega_2-2s-k)}; q^2)_s$ (the general solution is obtained by multiplying each ϕ_k with a fixed 1-periodic function). With this choice of ϕ_k, we know that (6.8) holds for $x = \alpha$, $x = \gamma$, $x = \alpha\delta$ and $x = \gamma\beta$. Since it is clear from (6.6) and (6.7) that $\pi(\gamma)$ and $\pi(\alpha)$ are injective, we can conclude that (6.8) holds also for $x = \delta$ and $x = \beta$, and thus for any $x \in \mathcal{F}_R(\mathrm{SL}(2))$. Since we have already observed that the eigenvectors $v(y; p)$ form an $M_{\mathfrak{h}^*}$-basis of $\mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2}$, the following theorem is now clear.

Theorem 6.2. *Assuming that $\omega_1 + \omega_2 \notin \mathbb{Z}_{\leq 0}$, the decomposition (6.1) holds as an equivalence of dynamical representations. Moreover,*

$$Ce_k = q^{s(k-\lambda)}(q^{2(\lambda+2-\omega_1-\omega_2-2s-k)}; q^2)_s\, v(s; s + k),$$

with the notation of Proposition 6.1, defines an intertwiner $C : \mathcal{H}^{\omega_1+\omega_2+2s} \to \mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2}$.

We can now interpret Proposition 6.1 as stating that the q-Racah polynomials are Clebsch–Gordan coefficients for the representations π^ω. In analogy with corepresentations, we will write

$$Ce_k = \sum_{l+m=s+k} C^{\omega_1+\omega_2+2s,\omega_1\omega_2}_{k,lm}(e_l \otimes e_m),$$

where, writing $L = l + m = k + s$,

$$C^{\omega_1+\omega_2+2s,\omega_1\omega_2}_{k,lm}(\lambda) = (-1)^m q^{s(k-\lambda)+2m(\omega_1+\omega_2)+m(3L-1)}$$

(6.9)
$$\times \frac{(q^{-2\lambda}, q^{-2(\omega_1+L-1)}, q^{-2L}; q^2)_m (q^{2(\lambda+2-\omega_1-\omega_2-s-L)}; q^2)_s}{(q^2, q^{2\omega_2}, q^{2(\omega_1+\omega_2+L-\lambda-1)}; q^2)_m}$$
$$\times R_m(\mu(k); q^{-2(\lambda+1)}, q^{2(\omega_2-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2).$$

For later use we note the alternative expression

$$C^{\omega_1+\omega_2+2s,\omega_1\omega_2}_{k,lm}(\lambda) = (-1)^s q^{s(\lambda+1-L)+2l(1-\omega_1-\omega_2-L)-lm}\begin{bmatrix} L \\ m \end{bmatrix}_{q^2}$$

(6.10)
$$\times \frac{(q^{-2\lambda}, q^{-2(\omega_1+L-1)}; q^2)_m (q^{2\omega_2}; q^2)_L (q^{2(\omega_1+\omega_2-\lambda-1+m+L)}; q^2)_l}{(q^{-2\lambda}, q^{-2(\omega_1+L-1)}; q^2)_k (q^{2\omega_2}; q^2)_m (q^{2\omega_2}; q^2)_s}$$
$$\times R_l(\mu(k); q^{2(\lambda-\omega_1-\omega_2+1-2L)}, q^{2(\omega_1-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2),$$

which follows by combining (6.9) and (2.22). Finally we remark that if we instead use (2.21) we obtain

$$
C^{\omega_1+\omega_2+2s,\omega_1\omega_2}_{k,lm}(\lambda) = q^{s(k-\lambda)-lm}(q^{2(\lambda+2-\omega_1-\omega_2-s-L)};q^2)_s \begin{bmatrix} L \\ m \end{bmatrix}_{q^2}
$$
$$
\times R_s(\mu(m);q^{2(\omega_2-1)},q^{2(\omega_1-1)},q^{-2(L+1)},q^{2(L+\omega_2-\lambda-1)};q^2).
$$

Comparing with (5.19), we see that the Clebsch–Gordan coefficients for $\mathcal{H}^{\omega_1}\bar{\otimes}\mathcal{H}^{\omega_2}$ can be obtained from those of $V_M\widehat{\otimes}V_N$ by the formal replacements $\omega_1 = -N$, $\omega_2 = -M$. This is analogous to the connection between highest weight representations of (Lie or quantum) $\mathrm{SU}(2)$ and $\mathrm{SU}(1,1)$.

6.2. **The pentagonal identity.** The classical and quantum $6j$-symbols satisfy the pentagonal or Biedenharn–Elliott identity. For the $6j$-symbols of $\mathcal{U}_q(\mathfrak{su}(2))$ [KR], this identity can be written in terms of q-Racah polynomials in many ways, such as

$$
(6.11) \quad \frac{(aq,q^{-m_1-m_2},bdq^{1-m_3};q)_{k_1}(abq^{2k_1+2};q)_{k_2}(bcq^2;q)_{k_1+k_2}}{(bq,bcdq^2,q^{-m_1-m_2-m_3};q)_{k_1}(cq;q)_{k_2}(aq;q)_{k_1+k_2}}(bq^{1+k_1})^{-k_2}
$$
$$
\times R_{k_1}(\mu(m_1);a,b,q^{-(m_1+m_2+1)},dq^{-m_3};q)
$$
$$
\times R_{k_2}(\mu(m_1+m_2-k_1);abq^{2k_1+1},c,q^{-(m_1+m_2+m_3-k_1+1)},bdq^{1+k_1};q)
$$
$$
= \sum_{l=0}^{\min(k_1+k_2,m_2+m_3)} \frac{(bcq,q^{-k_1-k_2},q^{-m_2-m_3},abcq^{k_1+k_2+2},bq,bcdq^{2+m_1};q)_l}{(q,bcq^{k_1+k_2+2},q^{-m_1-m_2-m_3},q^{-k_1-k_2}a^{-1},cq,bcdq^2;q)_l}\frac{1-bcq^{2l+1}}{1-bcq}
$$
$$
\times (abq^{1+m_1})^{-l}R_{k_1}(\mu(l);b,a,q^{-(k_1+k_2+1)},bcq^{k_1+k_2+1};q)
$$
$$
\times R_l(\mu(m_2);b,c,q^{-(m_2+m_3+1)},bdq^{1+m_1};q)
$$
$$
\times R_{k_1+k_2-l}(\mu(m_1);a,bcq^{2l+1},q^{-(m_1+m_2+m_3-l+1)},dq^{-l};q).
$$

From the viewpoint of special functions, (6.11) is a master identity which contains many classical results as limit cases, including various convolution, linearization and addition formulas for orthogonal polynomials. We will be concerned with the case $a = q^{-N-1}$, $N \in \mathbb{Z}_{\geq 0}$. If $N < k_1 + k_2$, there is then a singularity which must be removed by multiplying with $(aq;q)_{k_1+k_2}$ and interpreting

$$
\frac{(aq;q)_{k_1+k_2}}{(q^{-k_1-k_2}a^{-1};q)_l} = (-1)^l q^{-\binom{l}{2}+l(k_1+k_2-N-1)}(q^{-N};q)_{k_1+k_2-l},
$$

so that the summation can be restricted to

$$
\max(0,k_1+k_2-N) \leq l \leq \min(k_1+k_2,m_1+m_2).
$$

From the interpretation in terms of $\mathcal{U}_q(\mathfrak{su}(2))$, one only obtains (6.11) for discrete values of the 9 free parameters (not counting q). It can be extended to continuous values of a, b, c, d by working instead with $\mathcal{U}_q(\mathfrak{su}(1,1))$. The equations (5.23) and (5.24) are instances of (6.11) with 2 continuous parameters. In this section we will obtain (6.11) with 3 continuous parameters using our interpretation of q-Racah polynomials as Clebsch–Gordan coefficients of dynamical representations. We point out that an extension of (6.11) to the case of 9 continuous parameters was obtained in

[KV2], again using a quantum algebraic interpretation. This involves not necessarily terminating $_8W_7$-series.

To obtain the pentagonal identity, we corepresent the intertwining property (6.8) by applying it to $x = t_{kj}^N$ and acting on $e_m \in \mathcal{H}^{\omega_1+\omega_2+2s}$:

$$(6.12) \qquad C\pi^{\omega_1+\omega_2+2s}(t_{kj}^N)e_m = \pi^\omega(t_{kj}^N)Ce_m.$$

The left-hand side of this identity is

$$CT_{kjm}^{\omega_1+\omega_2+2s,N}e_{m+j-k} = \sum_{x+y=s+m+j-k} C_{m+j-k,xy}^{\omega_1+\omega_2+2s,\omega_1\omega_2}T_{kjm}^{\omega_1+\omega_2+2s,N}(e_x \otimes e_y),$$

where $T_{kjm}^{\omega N}$ is the function from Proposition 4.5. The right-hand side of (6.12) is

$$\sum_{l=0}^N \pi^{\omega_1}(t_{kl}^N) \otimes \pi^{\omega_2}(t_{lj}^N) \sum_{x+y=s+m} e_x \otimes C_{m,xy}^{\omega_1+\omega_2+2s,\omega_1\omega_2}e_y$$

$$= \sum_{l=0}^N \sum_{x+y=s+m} T_{klx}^{\omega_1 N}e_{x+l-k} \otimes T_{ljy}^{\omega_2 N}(T_{N-2j}C_{m,xy}^{\omega_1+\omega_2+2s,\omega_1\omega_2})e_{y+j-l}$$

$$= \sum_{l=0}^N \sum_{x+y=s+m} (T_{-\omega_2-2(y+j-l)}T_{klx}^{\omega_1 N})T_{ljy}^{\omega_2 N}(T_{N-2j}C_{m,xy}^{\omega_1+\omega_2+2s,\omega_1\omega_2})(e_{x+l-k} \otimes e_{y+j-l}).$$

Identifying the coefficient of $e_x \otimes e_y$, we obtain

$$(6.13) \quad C_{m+j-k,xy}^{\omega_1+\omega_2+2s,\omega_1\omega_2}(\lambda)T_{kjm}^{\omega_1+\omega_2+2s,N}(\lambda)$$

$$= \sum_{l=\max(0,j-y)}^{\min(N,x+k)} C_{m,x+k-l,y+l-j}^{\omega_1+\omega_2+2s,\omega_1\omega_2}(\lambda+N-2j)T_{kl,x+k-l}^{\omega_1 N}(\lambda-\omega_2-2y)T_{lj,y+l-j}^{\omega_2 N}(\lambda),$$

where $k + x + y = s + m + j$.

To identify (6.13) as a special case of (6.11), we first plug in the expressions from Proposition 4.5 and (6.10). We then transform the q-Racah polynomials coming from T_{kjm} and $T_{lj,y+l-j}$ using (2.20) and the one coming from $T_{kl,x+k-l}$ using (2.21). Finally we replace l by $x + k - l$ in the summation. As the patient reader can verify, we obtain (6.11) in base q^2 with

$$(k_1, k_2, m_1, m_2, m_3, a, b, c, d)$$
$$\mapsto (k, x, j, m, s, q^{-2(N+1)}, q^{2(\lambda-\omega_1-\omega_2+N+1-2j-2m-2s)}, q^{2(\omega_1-1)}, q^{-2(\lambda+1+N-j-m-s)}).$$

In the case of the group SU(2), (3.5) for $\Delta(t_{kk}^{2k})$ (the spherical case) is the classical addition formula for Legendre polynomials. In [K2], Koornwinder showed that for $\mathcal{F}_q(\mathrm{SU}(2))$, one may obtain an addition formula for little q-Legendre polynomials by evaluating the corresponding identity in a tensor product of infinite-dimensional representations. This became the starting point for much work on quantum groups and q-special functions, cf. [Ko2]. In view of Remark 4.2 one might expect that Koornwinder's formula is a limit case of the spherical case of (6.13), and thus of the Biedenharn–Elliott identity (6.11). However, since the formal limit $\mathcal{F}_R(\mathrm{SL}(2)) \rightarrow$

$\mathcal{F}_q(\mathrm{SL}(2))$ involved here does not preserve the coproduct, this is not at all clear a priori. Nevertheless, such a limit transition exists, as will be explained below. We stress that the general case of (6.11) had not yet appeared in the literature when [K2] was published.

To rewrite (6.13) in a form similar to the addition formula of [K2], we must express the functions $T_{kjm}^{\omega N}$ in terms of appropriate Askey–Wilson polynomials. Namely, when $k \leq j$, $k + j \leq N$ we use the expression (4.5) for $T = T_{kjm}^{\omega N}$. When $k \leq j$, $k + j \geq N$ we write

$$T_{kjm}^{\omega N}(\lambda) = (-1)^{N-k} q^{(N-j)(2\lambda-\omega+3+N+k-4j-m)+k(1-m)-N}$$
$$\times \frac{(q^{2(\lambda-\omega+2+N-2j-m)}; q^2)_{k+j-N}}{(q^2, q^{2(\lambda+1-j)}; q^2)_{N-j}(q^{2(\lambda-\omega+2+N-2j-2m)}; q^2)_k}$$
$$\times p_{N-j}\big(\tfrac{q^{\omega-1}+q^{1-\omega}}{2}; q^{1+\omega+2(m+j-k)}, q^{1-\omega-2m}, q^{3-\omega+2(\lambda+k-j-m)}, q^{\omega-1-2(\lambda+N-2j-m)}; q^2\big).$$

When $j \leq k$, $k + j \leq N$, we write

$$T_{kjm}^{\omega N}(\lambda) = (-1)^k q^{k(2\lambda+3+N-j-m)-j(\omega+1+N+3m)+mN}$$
$$\times \frac{(q^{2(\lambda+1+k-j-m)}; q^2)_{N-k-j}(q^{-2m}, q^{-2(m+\omega-1)}; q^2)_{k-j}}{(q^2; q^2)_j(q^{2(\lambda+1-j)}; q^2)_{N-j}(q^{2(\lambda-\omega+2+N-2j-2m)}; q^2)_k}$$
$$\times p_j\big(\tfrac{q^{\omega-1}+q^{1-\omega}}{2}; q^{1+\omega+2m}, q^{1-\omega-2(m+j-k)}, q^{3-\omega+2(\lambda+N-2j-m)}, q^{\omega-1-2(\lambda+k-j-m)}; q^2\big),$$

while, finally, if $j \leq k$, $N \leq k + j$ we write

$$T_{kjm}^{\omega N}(\lambda) = (-1)^{N-j} q^{(N-j)(2\lambda-\omega+2+N+k-4j-m)+(k-j)(\omega+1+j-k+2m)-mk}$$
$$\times \frac{(q^2; q^2)_k(q^{-2m}, q^{-2(m+\omega-1)}; q^2)_{k-j}(q^{2(\lambda-\omega+2+N-2j-m)}; q^2)_{k+j-N}}{(q^2; q^2)_j(q^2, q^{2(\lambda+1-j)}; q^2)_{N-j}(q^{2(\lambda-\omega+2+N-2j-2m)}; q^2)_k}$$
$$\times p_{N-k}\big(\tfrac{q^{\omega-1}+q^{1-\omega}}{2}; q^{1+\omega+2m}, q^{1-\omega-2(m+j-k)}, q^{3-\omega+2(\lambda+k-j-m)}, q^{\omega-1-2(\lambda+N-2j-m)}; q^2\big).$$

These expressions can be obtained from Proposition 4.5 using transformation formulas from [GR], or directly from Theorem 3.5. If we insist on using these expressions, we must split the sum in (6.13) into five parts, according to whether l is smaller or larger than j, k, $N - j$ and $N - k$. We will write this out explicitly only in the spherical case, when the four splitting points agree.

Thus we let $j = k$, $N = 2k$ in (6.13). After rewriting the sum as

$$\sum_{l=\max(0,k-y)}^{\min(2k,x+k)} a_l = a_k + \sum_{l=1}^{\min(k,y)} a_{k-l} + \sum_{l=1}^{\min(k,x)} a_{k+l},$$

we express the functions $T_{kjm}^{\omega N}$ as indicated above and the Clebsch–Gordan coefficients using (6.10). This results in the identity

$$
q^{-k}(q^2, q^{2(\lambda-\omega_2+2-2y)}, q^{2(\omega_2-\lambda+2y)}; q^2)_k
$$
$$
\times R_x(\mu(m); q^{2(\lambda-\omega_1-\omega_2+1-2L)}, q^{2(\omega_1-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2)
$$
$$
\times p_k(\Omega; q^{1+\omega_1+\omega_2+2L}, q^{1-\omega_1-\omega_2-2L}, q^{3-\omega_1-\omega_2+2(\lambda-L)}, q^{\omega_1+\omega_2-1-2(\lambda-L)}; q^2)
$$
$$
= R_x(\mu(m); q^{2(\lambda-\omega_1-\omega_2+1-2L)}, q^{2(\omega_1-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2)
$$
$$
\times p_k(\Omega_1; q^{1+\omega_1+2x}, q^{1-\omega_1-2x}, q^{3-\omega_1+2(\lambda-\omega_2-x-2y)}, q^{\omega_1-1-2(\lambda-\omega_2-x-2y)}; q^2)
$$
$$
\times p_k(\Omega_2; q^{1+\omega_2+2y}, q^{1-\omega_2-2y}, q^{3-\omega_2+2(\lambda-y)}, q^{\omega_2-1-2(\lambda-y)}; q^2)
$$
$$
+ \sum_{l=1}^{\min(k,y)} \frac{(q^2;q^2)_{k+l}(q^2;q^2)_y}{(q^2;q^2)_{k-l}(q^2;q^2)_{y-l}} \frac{1-q^{2(\lambda-\omega_2+1-2y+2l)}}{1-q^{2(\lambda-\omega_2+1-2y)}} q^{l(\omega_1+3\omega_2-2\lambda-4+2k+2x+4y)}
$$
$$
\times \frac{(q^{2(1-\omega_2-y)}, q^{2(\lambda-\omega_1-\omega_2+2-x-2y)}, q^{2(\lambda-\omega_2+1-x-2y)}, q^{2(\lambda-\omega_2+1-k-2y)}; q^2)_l}{(q^{2(\lambda-\omega_2+2+k-2y)}; q^2)_l}
$$
$$
\times R_{x+l}(\mu(m); q^{2(\lambda-\omega_1-\omega_2+1-2L)}, q^{2(\omega_1-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2)
$$
$$
\times p_{k-l}(\Omega_1; q^{1+\omega_1+2(x+l)}, q^{1-\omega_1-2x}, q^{3-\omega_1+2(\lambda-\omega_2-x-2y+l)}, q^{\omega_1-1-2(\lambda-\omega_2-x-2y)}; q^2)
$$
$$
\times p_{k-l}(\Omega_2; q^{1+\omega_2+2y}, q^{1-\omega_2-2(y-l)}, q^{3-\omega_2+2(\lambda-y+l)}, q^{\omega_2-1-2(\lambda-y)}; q^2)
$$
$$
+ \sum_{l=1}^{\min(k,x)} \frac{(q^2;q^2)_{k+l}(q^2;q^2)_x}{(q^2;q^2)_{k-l}(q^2;q^2)_{x-l}} \frac{1-q^{2(\omega_2-\lambda-1+2y+2l)}}{1-q^{2(\omega_2-\lambda-1+2y)}} q^{l(\omega_1-\omega_2+2\lambda+2k-2y)}
$$
$$
\times \frac{(q^{-2(\lambda-y)}, q^{2(1-\omega_1-x)}, q^{2(\omega_2-\lambda-1+y)}, q^{2(\omega_2-\lambda-1+2y-k)}; q^2)_l}{(q^{2(\omega_2-\lambda+k+2y)}; q^2)_l}
$$
$$
\times R_{x-l}(\mu(m); q^{2(\lambda-\omega_1-\omega_2+1-2L)}, q^{2(\omega_1-1)}, q^{-2(L+1)}, q^{-2(\omega_1+\omega_2+L-1)}; q^2)
$$
$$
\times p_{k-l}(\Omega_1; q^{1+\omega_1+2x}, q^{1-\omega_1-2(x-l)}, q^{3-\omega_1+2(\lambda-\omega_2-x-2y)}, q^{\omega_1-1-2(\lambda-\omega_2-x-2y-l)}; q^2)
$$
$$
\times p_{k-l}(\Omega_2; q^{1+\omega_2+2(y+l)}, q^{1-\omega_2-2y}, q^{3-\omega_2+2(\lambda-y)}, q^{\omega_2-1-2(\lambda-y-l)}; q^2),
$$

where $L = x + y = s + m$ and we write

$$
\Omega_1 = \frac{q^{\omega_1-1} + q^{1-\omega_1}}{2}, \quad \Omega_2 = \frac{q^{\omega_2-1} + q^{1-\omega_2}}{2}, \quad \Omega = \frac{q^{\omega_1+\omega_2+2s-1} + q^{1-\omega_1-\omega_2-2s}}{2}.
$$

This identity generalizes Koornwinder's addition formula to the level of Askey–Wilson polynomials, but is itself a special case of (6.11). It is also a special case of an identity obtained in [Ko2], using the non-dynamical quantum group and twisted primitive elements. Note that since $\mu(m) = 2q^{1-2L-\omega_1-\omega_2}\Omega$, we may view it as a linearization formula, expanding a product of two polynomials in $\mu(m)$ into q-Racah polynomials.

To obtain Koornwinder's formula as a limit case (cf. also [Ko2, Remark 5.2]) , we fix k, x and m. We then let $y, s, L, \lambda, \omega_1 + \omega_2 + L, -\omega_2 - L \to \infty$ in such a way that $L = x + y = s + m$ and $\lambda - L \to z$ for some constant z. Using limit transitions from [KS], or directly from the definitions, one may check that both the q-Racah and the

Askey–Wilson polynomials tend to little q-Jacobi polynomials, defined by (cf. [GR])

$$p_n(x; a, b; q) = {}_2\phi_1 \left[\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx \right].$$

In the limit, one obtains the identity

$$\begin{aligned}
p_x(q^{2m}; & q^{2z}, 0; q^2)\, p_k(q^{2m}; 1, 1; q^2) \\
&= p_x(q^{2m}; q^{2z}, 0; q^2)\, p_k(q^{2x}; 1, 1; q^2)\, p_k(q^{2(x+z)}; 1, 1; q^2) \\
&\quad + \sum_{l=1}^{k} \frac{(q^2; q^2)_{k+l}(q^{2(1+z)}; q^2)_{x+l}}{(q^2; q^2)_{k-l}(q^{2(1+z)}; q^2)_x (q^2; q^2)_l^2} q^{2l(x+l-k)} \\
&\qquad \times p_{x+l}(q^{2m}; q^{2z}, 0; q^2)\, p_{k-l}(q^{2x}; q^{2l}, q^{2l}; q^2)\, p_{k-l}(q^{2(x+z)}; q^{2l}, q^{2l}; q^2) \\
&\quad + \sum_{l=1}^{\min(k,x)} \frac{(q^2; q^2)_{k+l}(q^2; q^2)_x}{(q^2; q^2)_{k-l}(q^2; q^2)_{x-l}(q^2; q^2)_l^2} q^{2l(x+z-k+1)} \\
&\qquad \times p_{x-l}(q^{2m}; q^{2z}, 0; q^2)\, p_{k-l}(q^{2(x-l)}; q^{2l}, q^{2l}; q^2)\, p_{k-l}(q^{2(x+z-l)}; q^{2l}, q^{2l}; q^2).
\end{aligned}$$

For $z \in \mathbb{Z}_{\geq 0}$, this is Koornwinder's formula.

7. The Haar functional

In this section we will show that there is a natural Haar functional on $\mathcal{F}_R(\mathrm{SL}(2))$, and that it can be identified with a special case of the Askey–Wilson measure.

To motivate our definition, note that the Haar functional on a compact group can be obtained as the projection from the regular representation to the isotypic subspace containing the trivial representation. Since the trivial representation occurs with multiplicity one, the range of the Haar functional can be identified with $\mathbb{C}$. In the present case, cf. Remark 3.7, the trivial representation occurs with infinite multiplicity, and the corresponding isotypic component is $\mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*}) \simeq M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$. The projection onto this space is

(7.1) $$h(f(\lambda)g(\mu)t_{kj}^N) = f(\lambda)g(\mu)\,\delta_{0N}, \quad 0 \leq j, k \leq N.$$

By Proposition 3.6, this defines a $\mathbb{C}$-linear map $\mathcal{F}_R(\mathrm{SL}(2)) \to \mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*})$, which is an $\mathfrak{h}$-prealgebra homomorphism, cf. §5.1. We call this map the *Haar functional* on $\mathcal{F}_R(\mathrm{SL}(2))$.

We define a *left-invariant integral* on an $\mathfrak{h}$-bialgebroid A to be an $\mathfrak{h}$-prealgebra homomorphism $h : A \to \mu_l^A(M_{\mathfrak{h}^*})\mu_r^A(M_{\mathfrak{h}^*}) \subseteq A$ such that, under the identifications (2.4),

$$h = (\mathrm{id} \otimes \varepsilon \circ h) \circ \Delta.$$

If this condition is replaced with

$$h = (\varepsilon \circ h \otimes \mathrm{id}) \circ \Delta$$

we speak of a *right-invariant integral*. If $a = \sum_i a_i' \otimes a_i''$, and we write $\mu_l(f) = f(\lambda)$, $\mu_r(f) = f(\mu)$, then left-invariance means that

$$h(a)(\lambda, \mu) = \sum_i h(a_i'')(\mu, \mu)\, a_i'$$

and right-invariance that

$$h(a)(\lambda, \mu) = \sum_i h(a_i')(\lambda, \lambda)\, a_i''.$$

These definitions are motivated by the following fact.

Proposition 7.1. *The Haar functional* (7.1) *is both the unique left-invariant and the unique right-invariant integral on* $\mathcal{F}_R(\mathrm{SL}(2))$ *such that* $h(1) = 1$.

Proof. Let h be a left-invariant integral on $\mathcal{F}_R(\mathrm{SL}(2))$. Then $h(f(\lambda)g(\mu)t_{kj}^N) = f(\lambda)g(\mu)h_{kj}^N(\lambda, \mu)$ for some $h_{kj}^N \in M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$. The left-invariance of h means that

$$h_{kj}^N(\lambda, \mu) = \sum_{l=0}^N h_{lj}^N(\mu, \mu)t_{kl}^N.$$

By Proposition 3.6, this implies that $h_{kj}^N = 0$ unless $N = 0$, in which case the normalizing condition shows that h is given by (7.1). For the right-invariance, the proof is similar. $\qquad\square$

We will now obtain the Schur orthogonality relations for matrix elements. These are most elegantly discussed in terms of contragredient corepresentations, but to save space we give a direct proof using results obtained above.

Theorem 7.2. *In the notation above, the following Schur orthogonality relations are valid:*

$$h(t_{jk}^M(t_{lm}^N)^*) = \delta_{MN}\,\delta_{jl}\,\delta_{km}\, q^{2(M-k)}\frac{1-q^2}{1-q^{2(M+1)}}$$

$$\times \frac{(q^2, q^{-2\lambda}; q^2)_j (q^2; q^2)_{M-j}(q^{-2(\mu+1+M-k)}; q^2)_{M-k}}{(q^{-2(\lambda+M+1-j)}; q^2)_j(q^2; q^2)_k(q^2, q^{-2(\mu+M-2k)}; q^2)_{M-k}}.$$

Proof. Applying h to (5.22) gives

$$(7.2)\quad h(t_{kx}^M t_{ly}^N) = \delta_{MN}\,\delta_{k+l,M}\,\delta_{x+y,M}\frac{\Gamma_x^M(\mu)\Gamma_y^M(\mu+M-2x)}{Z_M\Gamma_0^0(\mu)}C_{kl,0}^{MM,0}(\lambda)C_{xy,0}^{MM,0}(\mu)$$

$$= \delta_{MN}\,\delta_{k+l,M}\,\delta_{x+y,M}(-q)^{l+y}\frac{(q^2, q^{-2\lambda}; q^2)_k(q^2; q^2)_l(q^{-2(\mu+1+y)}; q^2)_y}{(q^{-2(\lambda+1+l)}; q^2)_k(q^2; q^2)_x(q^2, q^{-2(\mu+y-x)}; q^2)_y}\frac{1-q^2}{1-q^{2(M+1)}},$$

where we inserted the expressions (5.17) for the Clebsch–Gordan coefficients. Next we observe that

$$(7.3)\qquad\qquad (t_{kj}^N)^* = (-q)^{k-j}t_{N-k,N-j}^N.$$

This follows easily from Proposition 3.2, and can also be proved without using explicit expressions for the matrix elements, similarly to the proof of Proposition 3.12. Combining (7.2) and (7.3) completes the proof. $\qquad\square$

Note that h vanishes outside $\mathcal{F}_R(\mathrm{SL}(2))_{00}$, which can be identified with the algebra of polynomials in Ξ (cf. Lemma 3.3) over the meromorphic functions in λ and μ. It

is natural to seek a family of measures $dm_{\lambda\mu}$ so that

$$(7.4) \qquad\qquad h(p(\Xi)) = \int p(x)\, dm_{\lambda\mu}(x)$$

for any such polynomial p. By Proposition 3.6, it is enough to do this for t_{kk}^{2k}, $k \in \mathbb{Z}_{\geq 0}$. It follows from Theorem 3.5 that

$$t_{kk}^{2k} = N_k\, p_k(\tfrac{1}{2}\,\Xi),$$

with $N_k = q^{k(\lambda-\mu+1)}/(q^2, q^{-2\mu}; q^2)_k$, $p_k(x) = p_k(x; q^{\mu-\lambda+1}, q^{\lambda-\mu+1}, q^{\lambda+\mu+3}, q^{-\lambda-\mu-1}; q^2)$. For $\lambda, \mu \notin \mathbb{Z}$, the polynomials $\{p_k\}_{k=0}^{\infty}$ form an orthogonal system with respect to a moment functional, which is given by integration with respect to an explicitly known (not necessarily positive) measure, cf. [AW2]. Under additional conditions on λ and μ, e.g. $\lambda, \mu \in (j, j+1)$ for $j \in \mathbb{Z}$ or $\operatorname{Im} \lambda = \operatorname{Im} \mu = \pi/2 \log q$, the measure is positive. Assuming that $\lambda, \mu \notin \mathbb{Z}$, we let $dm_{\lambda\mu}$ be the orthogonality measure, rescaled and normalized so that

$$C_k\, \delta_{kl} = \int p_k(\tfrac{x}{2})\, p_l(\tfrac{x}{2})\, dm_{\lambda\mu}(x)$$

with $C_0 = 1$. Then, in particular, (7.4) is satisfied. Therefore, the Haar measure on $\mathcal{F}_R(\mathrm{SL}(2))$ can be identified with the orthogonality measure for a two-parameter family of Askey–Wilson polynomials. A similar interpretation is obtained by Koornwinder in the non-dynamical case [K3], using twisted primitive elements.

Let us now combine (7.4) with the case of Theorem 7.2 when h is applied to an element of $\mathcal{F}_R(\mathrm{SL}(2))_{00}$. It will be no restriction to assume that we are in the first parameter domain of Theorem 3.5. Thus we consider the element $t_{j,j+k}^{2j+k+m}(t_{l,l+k}^{2l+k+m})^*$, where $j, k, l, m \in \mathbb{Z}_{\geq 0}$. Using Theorem 3.5, Lemma 3.1 and Lemma 3.4, we can write

$$t_{j,j+k}^{2j+k+m}(t_{l,l+k}^{2l+k+m})^* = (-1)^m q^{(j+l)(\lambda-\mu+1+k)-m(2\mu+1+m)} \frac{(q^{-2\mu}; q^2)_{k-m}}{(q^{-2\mu}; q^2)_{j+k}(q^{-2\mu}; q^2)_{l+k}}$$

$$\times \frac{(q^2; q^2)_{j+k+m}(q^2; q^2)_{l+k+m}}{(q^2; q^2)_{j+k}(q^2; q^2)_{j+m}(q^2; q^2)_{l+k}(q^2; q^2)_{l+m}} \frac{(q^{2(\lambda+2)}; q^2)_{k+m}}{(q^{2(\lambda+2)}; q^2)_{j+k+m}(q^{2(\lambda+2)}; q^2)_{l+k+m}}$$

$$\times h_k(\tfrac{1}{2}\Xi, q^{\lambda-\mu+1}; q^2)\, h_m(\tfrac{1}{2}\Xi, q^{\lambda+\mu+3}; q^2)\, p_j^{(k,m)}(\tfrac{1}{2}\Xi)\, p_l^{(k,m)}(\tfrac{1}{2}\Xi),$$

where we use the notation (2.15) if $k < m$ and write (cf. (3.14))

$$p_j^{(k,m)}(x) = p_j(x; q^{\mu-\lambda+1}, q^{\lambda-\mu+1+2k}, q^{\lambda+\mu+3+2m}, q^{-\lambda-\mu-1}; q^2).$$

Combining this with Theorem 7.2 and (7.4), we get after simplifications

$$\int p_j^{(k,m)}(\tfrac{x}{2})\, p_l^{(k,m)}(\tfrac{x}{2})\, h_k(\tfrac{x}{2}, q^{\lambda-\mu+1}; q^2)\, h_m(\tfrac{x}{2}, q^{\lambda+\mu+3}; q^2)\, dm_{\lambda\mu}(x)$$

$$= \delta_{jl} \frac{1-q^2}{1-q^{2(2j+k+m+1)}} \frac{(q^2; q^2)_j(q^2; q^2)_{j+k}(q^2; q^2)_{j+m}}{(q^2; q^2)_{j+k+m}}$$

$$\times (q^{-2\lambda}; q^2)_j(q^{2(\lambda+2)}; q^2)_{j+k+m}(q^{-2\mu}; q^2)_{j+k}(q^{2(\mu+2)}; q^2)_{j+m}.$$

Thus we obtain the orthogonality of a four-parameter family of Askey–Wilson polynomials. The above expression agrees with the known explicit formulas for the orthogonality measure and the norm.

APPENDIX 1. PROPERTIES OF THE ANTIPODE

In this appendix we prove Proposition 2.2 and Lemma 2.9, though we leave many details to the reader. Recall that when proving the corresponding statements for Hopf algebras (cf. [Ka] for a detailed exposition) it is convenient to work with the convolution product

$$(A.1) \qquad \phi \star \psi = m^B \circ (\phi \otimes \psi) \circ \Delta^A$$

on $\mathrm{Hom}_{\mathbb{C}}(A, B)$, where A is a coalgebra and B an algebra. We will use the analogous convolution when A is an $\mathfrak{h}$-coalgebroid (cf. §5.1) and B an $\mathfrak{h}$-algebra. We will write $H_l = H_l(A, B)$ etc. for the spaces

$$H_l = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(A, B); \ \phi(\mu_l^A(f)a) = \mu_l^B(f)\phi(a) \ \text{ for all } a \in A, \ f \in M_{\mathfrak{h}^*}\},$$

$$H_r = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(A, B); \ \phi(a\mu_r^A(f)) = \phi(a)\mu_r^B(f) \ \text{ for all } a \in A, \ f \in M_{\mathfrak{h}^*}\},$$

$$H_l^{\mathrm{op}} = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(A, B); \ \phi(a\mu_l^A(f)) = \mu_r^B(f)\phi(a) \ \text{ for all } a \in A, \ f \in M_{\mathfrak{h}^*}\},$$

$$H_r^{\mathrm{op}} = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(A, B); \ \phi(\mu_r^A(f)a) = \phi(a)\mu_l^B(f) \ \text{ for all } a \in A, \ f \in M_{\mathfrak{h}^*}\}.$$

Note that, for $\mathfrak{h} \neq 0$, the convolution product (A.1) is not globally defined on $\mathrm{Hom}_{\mathbb{C}}(A, B) \times \mathrm{Hom}_{\mathbb{C}}(A, B)$, since $m \circ (\phi \otimes \psi)$ need not factor through relation (2.2). A sufficient condition for $\phi \star \psi$ to be well-defined is

$$\phi(\mu_r^A(f)a)\psi(b) = \phi(a)\psi(\mu_l^B(f)b), \quad a \in A_{\alpha\beta}, \ b \in A_{\beta\gamma}, \ f \in M_{\mathfrak{h}^*}.$$

Using this condition one proves the following lemma.

Lemma A.3. *The convolution $\star$ is well-defined on $H_r^{\mathrm{op}} \times H_l$ and on $H_r \times H_l^{\mathrm{op}}$. The associative law $(\phi \star \psi) \star \chi = \phi \star (\psi \star \chi)$ holds whenever both sides are well-defined.*

We now define $1_l = 1_l^{(A,B)}$, $1_r = 1_r^{(A,B)} \in \mathrm{Hom}_{\mathbb{C}}(A, B)$ by

$$1_l(a) = \mu_r^B(T_\alpha(\varepsilon^A(a)1)), \qquad 1_r(a) = \mu_l^B(\varepsilon^A(a)1), \quad a \in A_{\alpha\beta}.$$

These elements are functorial in the sense that if $\chi : A_1 \to A_2$ is an $\mathfrak{h}$-coalgebroid homomorphism and $\omega : B_1 \to B_2$ an $\mathfrak{h}$-prealgebra homomorphism, then

$$(A.2) \qquad 1_x^{(A_1,B)} = 1_x^{(A_2,B)} \circ \chi, \qquad 1_x^{(A,B_2)} = \omega \circ 1_x^{(A,B_1)}, \quad x = l, r.$$

One easily checks that $1_l \in H_r \cap H_l^{\mathrm{op}}$, $1_r \in H_l \cap H_r^{\mathrm{op}}$. In particular, the following lemma is meaningful. Note that it depends crucially on the dynamical shift in the definition of 1_l.

Lemma A.4. *The elements 1_l and 1_r satisfy*

$$1_l \star \phi = \phi, \qquad \psi \star 1_r = \psi, \quad \phi \in H_l^{\mathrm{op}}, \ \psi \in H_r^{\mathrm{op}}.$$

Proof. We write out the proof for 1_l, the case of 1_r being slightly easier. It suffices to evaluate both sides on $a \in A_{\alpha\beta}$. Let us write $\Delta(a) = \sum_i a'_i \otimes a''_i$, $\varepsilon(a'_i) = f_i T_{-\alpha}$. One then has

$$(A.3) \qquad (1_l \star \phi)(a) = \sum_i 1_l(a'_i)\phi(a''_i) = \sum_i \mu_r(T_\alpha f_i)\phi(a''_i).$$

On the other hand,

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(a) = \sum_i \varepsilon(a'_i) \otimes a''_i = \sum_i f_i T_{-\alpha} \otimes a''_i = \sum_i T_{-\alpha} \otimes \mu_l(f_i)a''_i,$$

using (2.2) in the last step. By the counit axioms this implies

$$a = \sum_i \mu_l(f_i)a''_i = \sum_i a''_i \mu_l(T_\alpha f_i),$$

using (2.1) and the fact that $f_i = 0$ unless $a'_i \in A_{\alpha\alpha}$, $a''_i \in A_{\alpha\beta}$. Applying ϕ to this identity and using that $\phi \in H_l^{\mathrm{op}}$ gives

$$\phi(a) = \sum_i \mu_r(T_\alpha f_i)\phi(\alpha''_i).$$

Comparing with (A.3) we see that $1_l \star \phi = \phi$. $\qquad\qquad\qquad\qquad\qquad \square$

We can now begin the proof of Proposition 2.2. The antipode axioms can be written as $S \in H_l^{\mathrm{op}} \cap H_r^{\mathrm{op}}$, $S \star \mathrm{id} = 1_l$, $\mathrm{id} \star S = 1_r$. If S and T are two such maps, it follows from the previous lemmas that

$$S = S \star 1_r = S \star (\mathrm{id} \star T) = (S \star \mathrm{id}) \star T = 1_l \star T = T.$$

This proves the uniqueness of the antipode.

It is easy to check that, for any $0 \neq f \in M_{\mathfrak{h}^*}$, the maps $S_f(a) = \mu_l(f^{-1})S(a\mu_r(f))$ and $S'_f(a) = S(\mu_l(f)a)\mu_r(f^{-1})$ satisfy the antipode axioms. By the uniqueness it follows that $S = S_f = S'_f$, which means that

$$(A.4) \qquad S(\mu_l(f)a) = S(a)\mu_r(f), \qquad S(a\mu_r(f)) = \mu_l(f)S(a), \quad a \in A, \ f \in M_{\mathfrak{h}^*}.$$

Together with (2.5), this implies (2.7).

To show the first identity of (2.8) one defines $\phi(a \otimes b) = S(b)S(a)$, $\psi(a \otimes b) = S(ab)$ and checks that they factor into maps $\phi \in H_l^{\mathrm{op}}(A\widehat{\otimes}A, A)$, $\psi \in H_r^{\mathrm{op}}(A\widehat{\otimes}A, A)$ satisfying

$$(A.5) \qquad\qquad\qquad m \star \phi = 1_r, \qquad \psi \star m = 1_l.$$

One needs (A.4) to show that ϕ factors through relation (5.1). It follows that

$$\phi = 1_l \star \phi = \psi \star m \star \phi = \psi \star 1_r = \psi.$$

Similarly one defines $\pi = \Delta \circ S$, $\rho = \sigma \circ (S \otimes S) \circ \Delta$ and checks that $\pi \in H_l^{\mathrm{op}}(A, A\widetilde{\otimes}A)$, $\rho \in H_r^{\mathrm{op}}(A, A\widetilde{\otimes}A)$,

$$(A.6) \qquad\qquad\qquad \Delta \star \pi = 1_r, \qquad \rho \star \Delta = 1_l,$$

which implies that $\pi = \rho$. One needs (2.7) to show that $\sigma \circ (S \otimes S)$ factors through (2.2), so that ρ is well-defined.

We write down the proof of (A.5) and (A.6) and leave the remaining details to the reader. In the case of (A.5), it suffices to evaluate both sides on $a \otimes b \in A_{\alpha\beta} \otimes A_{\gamma\delta}$. With notation as in (5.3), one then has

$$(m \star \phi)(a \otimes b) = \sum_{ij} m(a_i' \otimes b_j')\phi(a_i'' \otimes b_j'') = \sum_{ij} a_i'b_j'S(b_j'')S(a_i'')$$

$$= \sum_i a_i'(\mathrm{id} \star S)(b)S(a_i'') = \sum_i a_i'\,\mu_l(\varepsilon(b)1)S(a_i'') = \mu_l(T_{-\alpha}\varepsilon(b)1)(\mathrm{id} \star S)(a)$$

$$= \mu_l(T_{-\alpha}\varepsilon(b)1)\,\mu_l(\varepsilon(a)1) = \mu_l(\varepsilon(a)\varepsilon(b)1) = 1_r^{(A\widehat{\otimes}A,A)}(a \otimes b),$$

using in the penultimate step that $\varepsilon(a) \in M_{\mathfrak{h}^*}T_{-\alpha}$. Similarly,

$$(\psi \star m)(a \otimes b) = \sum_{ij} S(a_i'b_j')a_i''b_j'' = (S \star \mathrm{id})(ab) = 1_l^{(A,A)}(ab) = 1_l^{(A\widehat{\otimes}A,A)}(a \otimes b),$$

where we in the last step used (A.2) with $\chi = m$. The first equation of (A.6) follows from

$$(\Delta \star \pi)(a) = \sum_i \Delta(a_i')\Delta(S(a_i'')) = \Delta\left(\sum_i a_i'S(a_i'')\right) = \Delta(1_r^{(A,A)}(a)) = 1_r^{(A,A\widetilde{\otimes}A)}(a).$$

For the last equation we use notation such as $(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_i a_i^1 \otimes a_i^2 \otimes a_i^3$. Then (note that the symbols a_i^j have different meaning from equation to equation)

$$(\rho \star \Delta)(a) = \sum_i (\sigma(S \otimes S)\Delta)(a_i^1)\Delta(a_i^2) = \sum_i S(a_i^2)a_i^3 \otimes S(a_i^1)a_i^4$$

$$= \sum_i (S \star \mathrm{id})(a_i^2) \otimes S(a_i^1)a_i^3 = \sum_i \mu_r(T_{\beta_i}\varepsilon(a_i^2)1) \otimes S(a_i^1)a_i^3$$

$$= \sum_i 1 \otimes \mu_l(T_{\beta_i}\varepsilon(a_i^2)1)S(a_i^1)a_i^3 = \sum_i 1 \otimes S(a_i^1)\mu_l(\varepsilon(a_i^2)1)a_i^3$$

$$= 1 \otimes (S \star 1_r \star \mathrm{id})(a) = 1 \otimes (S \star \mathrm{id})(a) = 1 \otimes \mu_r(T_\alpha\varepsilon(a)1) = 1_l^{(A,A\widetilde{\otimes}A)}(a),$$

where we assume that in a three-fold tensor product $a_i^1 \in A_{\alpha\beta_i}$, $a_i^2 \in A_{\beta_i\gamma_i}$. In the sixth equality we used that, by (2.7), $S(a_i^1) \in A_{-\beta_i,-\alpha}$.

That $S(1) = 1$ is the case $a = 1$ of (2.6). Then (2.9) is obtained as the case $a = 1$ of (2.5). To prove that $\varepsilon \circ S = S^{D_{\mathfrak{h}}} \circ \varepsilon$, we write

$$\varepsilon(S(x)) = \varepsilon((1_l \star S)(x)) = \varepsilon\left(\sum_i 1_l^{(A,A)}(x_i')S(x_i'')\right) = \sum_i 1_l^{(A,D_{\mathfrak{h}})}(x_i')\,\varepsilon(S(x_i''))$$

$$= \sum_i T_\alpha(\varepsilon(x_i')1)\,\varepsilon(S(x_i'')) = \sum_i T_\alpha(\varepsilon(x_i')\varepsilon(S(x_i''))1)T_\beta = T_\alpha(\varepsilon((\mathrm{id} \star S)(x))1)T_\beta$$

$$= T_\alpha(\varepsilon(1_r^{(A,A)}(x))1)T_\beta = T_\alpha(1_r^{(A,D_{\mathfrak{h}})}(x)1)T_\beta = T_\alpha(\varepsilon(x)1)T_\beta = S^{D_{\mathfrak{h}}}(\varepsilon(x)),$$

where $x \in A_{\alpha\beta}$. In the fifth equality we used that, by (2.7), $\varepsilon(S(x_i'')) \in M_{\mathfrak{h}^*}T_\beta$.

We now turn to the proof of the second part of Proposition 2.2. Let $X \subseteq A$ and $S \in \mathrm{End}_{\mathbb{C}}(A)$ satisfy the conditions stated there. It is clear that (2.5) holds. Since

the relations (2.6) are linear in a, it suffices to show that if they hold for a and b, they hold for ab. For the relation $\mathrm{id} \star S = 1_r$, we write

$$(\mathrm{id} \star S)(ab) = \sum_{ij} a'_i b'_j S(a''_i b''_j) = \sum_{ij} a'_i b'_j S(b''_j) S(a''_i).$$

From the proof that $m \star \phi = 1_r$ given above we see that this equals $1_r(ab)$ if the relation holds for a and b. For $S \star \mathrm{id}$, the proof is similar.

Finally we turn to the proof of Lemma 2.9. By the uniqueness of the antipode, it suffices to check that $\tilde{S} = * \circ S^{-1} \circ *$ satisfies the antipode axioms. We write down the proof that $\tilde{S}$ satisfies the first equality of (2.6) and leave the remaining details to the reader. One has

$$m \circ (\mathrm{id} \otimes \tilde{S}) \circ \Delta = m \circ (* \otimes *) \circ (\mathrm{id} \otimes S^{-1}) \circ (* \otimes *) \circ \Delta$$
$$= * \circ m \circ \sigma \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta \circ *$$
$$= * \circ m \circ \sigma \circ (\mathrm{id} \otimes S^{-1}) \circ \sigma \circ (S \otimes S) \circ \Delta \circ S^{-1} \circ *$$
$$= * \circ m \circ (\mathrm{id} \otimes S) \circ \Delta \circ S^{-1} \circ *,$$

where we used (2.8) in the third step. Since S satisfies (2.6), it follows that

$$m \circ (\mathrm{id} \otimes \tilde{S}) \circ \Delta(a) = \mu_l(\varepsilon(S^{-1}(a^*))1)^* = \mu_l\left((S^{D_\flat})^{-1} \circ *^{D_\flat} \circ \varepsilon(a)1\right)^*$$
$$= \mu_l(\overline{\varepsilon(a)1})^* = \mu_l(\varepsilon(a)1),$$

where we used (2.8) and the $*$-structure axioms.

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